$$p_1 = 6e^3 \frac{x}{9-6D+D^2}$$ P.I is found by division.

$$\begin{array}{r|l} & x/9 + 2/27 \\ \hline 9-6D+D^2 & x \\ & x - 2/3 \\ \hline & 2/3 \\ & 2/3 \\ \hline & 0 \end{array}$$

$$p_1 = 6e^3\left(\frac{x}{9} + \frac{2}{27}\right) = \frac{2e^3}{9}\,(3x+2)$$

$$p_2 = \frac{7e^{-2x}}{D^2-6D+9}$$

$$p_2 = \frac{7e^{-2x}}{(-2)^2-6(-2)+9} = \frac{7e^{-2x}}{25}$$

$$p_3 = -\log 2 \cdot \frac{e^{0x}}{D^2-6D+9} = -\frac{\log 2}{0-0+9} = -\frac{\log 2}{9}$$

Complete solution : $y = y_c + p_1 + p_2 + p_3$

Thus $$y = (c_1 + c_2 x)e^{3x} + \frac{2e^3}{9}(3x+2) + \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$$

Note : If the first term in R.H.S of the equation is $6e^{3x}$ we have,

$$p_1 = \frac{6e^{3x}}{D^2-6D+9} = \frac{6e^{3x}}{3^2-6.3+9} \quad (Dr. = 0)$$

$$= 6 \cdot x\,\frac{e^{3x}}{2D-6} = 6x\frac{e^{3x}}{2.3-6} \quad (Dr. = 0)$$

$$p_1 = 6x^2\frac{e^{3x}}{2} = 3x^2 e^{3x}.$$

70. *Solve the differential equation* $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + \sin x + x$

>> We have, $(D^2 - 6D + 25)\, y = e^{2x} + \sin x + x$

A.E is given by $m^2 - 6m + 25 = 0$

On solving, $m = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$

$\therefore \quad y_c = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$

$$y_p = \frac{e^{2x}}{D^2 - 6D + 25} + \frac{\sin x}{D^2 - 6D + 25} + \frac{x}{D^2 - 6D + 25}$$

$$= p_1 + p_2 + p_3 \quad (say)$$

$$p_1 = \frac{e^{2x}}{D^2 - 6D + 25} = \frac{e^{2x}}{2^2 - 6(2) + 25} = \frac{e^{2x}}{17}$$

$$p_2 = \frac{\sin x}{D^2 - 6D + 25} \quad \text{Replace } D^2 \text{ by } -1^2 = -1$$

$$= \frac{\sin x}{24 - 6D} = \frac{\sin x}{6(4 - D)}$$

i.e., $$= \frac{(4 + D)\sin x}{6(16 - D^2)} = \frac{4 \sin x + \cos x}{6(17)}$$

$$p_2 = \frac{4 \sin x + \cos x}{102}$$

$$p_3 = \frac{x}{25 - 6D + D^2} \quad \text{P.I is found by division.}$$

$$\begin{array}{r|l} & x/25 + 6/625 \\ \hline 25 - 6D + D^2 & x \\ & x - 6/25 \\ \hline & \quad 6/25 \\ & \quad 6/25 \\ \hline & \quad 0 \end{array}$$

$$p_3 = \frac{x}{25} + \frac{6}{625} = \frac{1}{625}(25x + 6)$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2 + p_3$

Thus $$y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + \frac{e^{2x}}{17} + \frac{4 \sin x + \cos x}{102} + \frac{1}{625}(25x + 6)$$

2.6 Solution of simultaneous differential equations

Let us suppose that x and y are functions of an independent variable t connected by a system of first order equations with $D = d/dt$

$$f_1(D)x + f_2(D)y = \phi_1(t) \quad \ldots (1)$$

$$g_1(D)x + g_2(D)y = \phi_2(t) \quad \ldots (2)$$

We employ the elementary technique of solving a system of linear algebraic equations in cancelling either of the dependent variables. (x *or* y)

Operating (1) with $g_1(D)$ and (2) with $f_1(D)$, x cancels out by subtraction. We obtain a second order differential equation in y which can be solved. x can be obtained independently by cancelling y or by substituting the obtained $y(t)$ in a suitable equation.

WORKED PROBLEMS

71. *Solve :* $\frac{dx}{dt} + 2y = -\sin t$, $\frac{dy}{dt} - 2x = \cos t$

Taking $D = \frac{d}{dt}$ we have the system of equations :

$$Dx + 2y = -\sin t \quad \ldots (1)$$

$$-2x + Dy = \cos t \quad \ldots (2)$$

Multiply (1) by 2 and operate (2) by D

ie., $2Dx + 4y = -2 \sin t$

$-2Dx + D^2 y = -\sin t$

Adding these we get, $(D^2 + 4)y = -3 \sin t$

A.E. is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$\therefore \quad y_c = c_1 \cos 2t + c_2 \sin 2t$

$$y_p = \frac{-3\sin t}{D^2+4} = \frac{-3\sin t}{-1^2+4} = -\sin t$$

$$y = y_c + y_p$$

Hence, $\boldsymbol{y = c_1 \cos 2t + c_2 \sin 2t - \sin t}$ ... (3)

By considering $\frac{dy}{dt} - 2x = \cos t$, we get $x = \frac{1}{2}\left[\frac{dy}{dt} - \cos t\right]$

Hence, $x = \frac{1}{2}\left[\frac{d}{dt}(c_1 \cos 2t + c_2 \sin 2t - \sin t) - \cos t\right]$ by using (3).

$$= \frac{1}{2}[-2c_1 \sin 2t + 2c_2 \cos 2t - \cos t - \cos t]$$

$\therefore$ $\boldsymbol{x = -c_1 \sin 2t + c_2 \cos 2t - \cos t}$ ... (4)

Thus (3) & (4) represents the complete solution of the given system of equations.

---

72. *Solve :* $\frac{dx}{dt} - 7x + y = 0, \; \frac{dy}{dt} - 2x - 5y = 0$

>> Taking $D = \frac{d}{dt}$ we have the system of equations :

$(D-7)x + y = 0$ ... (1)

$-2x + (D-5)y = 0$ ... (2)

Multiply (1) by 2 and operate (2) by $(D-7)$

ie., $2(D-7)x + 2y = 0$

$-2(D-7)x + (D-5)(D-7)y = 0$

Adding these we have, $[(D-5)(D-7)+2]y = 0$ or $(D^2 - 12D + 37)y = 0$

A.E. is $m^2 - 12m + 37 = 0$ or $(m-6)^2 + 1 = 0 \Rightarrow (m-6) = \pm i$

$\therefore$ $m = 6 \pm i$

Hence $\boldsymbol{y = e^{6t}(c_1 \cos t + c_2 \sin t)}$ ... (3)

By considering $\frac{dy}{dt} - 2x - 5y = 0$ we get $x = \frac{1}{2}\left(\frac{dy}{dt} - 5y\right)$

$$x = \frac{1}{2}\left\{\frac{d}{dt}[e^{6t}(c_1\cos t + c_2\sin t)] - 5e^{6t}(c_1\cos t + c_2\sin t)\right\}$$

$$x = \frac{1}{2}\left\{e^{6t}(-c_1\sin t + c_2\cos t) + 6e^{6t}(c_1\cos t + c_2\sin t) - 5e^{6t}(c_1\cos t + c_2\sin t)\right\}$$

$$= \frac{1}{2}\left\{e^{6t}(-c_1\sin t + c_2\cos t) + e^{6t}(c_1\cos t + c_2\sin t)\right\}$$

$$x = \frac{1}{2}\left\{(c_1 + c_2)e^{6t}\cos t + (c_2 - c_1)e^{6t}\sin t\right\} \qquad \ldots(4)$$

Thus (3) and (4) represents the complete solution of the given system of equations.

---

73. *Solve :* $\frac{dy}{dx} + y = z + e^x,\quad \frac{dz}{dx} + z = y + e^x$

Taking $D = \frac{d}{dx}$ we have the system of equations

$$(D+1)y - z = e^x \qquad \ldots(1)$$

$$-y + (D+1)z = e^x \qquad \ldots(2)$$

Operating (1) by $(D+1)$ we have,

$$(D+1)^2 y - (D+1)z = (D+1)e^x = 2e^x$$

$$-y + (D+1)z = e^x$$

Adding these we get, $[(D+1)^2 - 1]y = 3e^x$ or $(D^2 + 2D)y = 3e^x$

A.E is $m^2 + 2m = 0$ or $m(m+2) = 0 \Rightarrow m = 0, -2$

$$y_c = c_1 + c_2 e^{-2x}$$

$$y_p = \frac{3e^x}{D^2 + 2D} = \frac{3e^x}{1^2 + 2\cdot 1} = e^x$$

$$y = y_c + y_p$$

$$y = c_1 + c_2 e^{-2x} + e^x \qquad \ldots(3)$$

Let us now consider $\frac{dy}{dx} + y = z + e^x$

$$\therefore \quad z = \frac{dy}{dx} + y - e^x$$

$$= \frac{d}{dx}(c_1 + c_2 e^{-2x} + e^x) + (c_1 + c_2 e^{-2x} + e^x) - e^x$$

$$= -2c_2 e^{-2x} + e^x + c_1 + c_2 e^{-2x}$$

$$z = c_1 - c_2 e^{-2x} + e^x \quad \ldots (4)$$

Thus, (3) and (4) represents the required solution.

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74. *Solve:* $\frac{dx}{dt} = 2x - 3y$, $\frac{dy}{dt} = y - 2x$ *given* $x(0) = 8$ *and* $y(0) = 3$

>> Taking $D = \frac{d}{dt}$ we have the system of equations

$$Dx = 2x - 3y \ ; \ Dy = y - 2x$$

or $$(D-2)x + 3y = 0 \quad \ldots (1)$$

$$2x + (D-1)y = 0 \quad \ldots (2)$$

Multiplying (1) by 2 and operating (2) by $(D-2)$ we get,

$$2(D-2)x + 6y = 0$$

$$2(D-2)x + (D-1)(D-2)y = 0$$

Subtracting we get $(D^2 - 3D - 4)y = 0$

A.E is $m^2 - 3m - 4 = 0$ or $(m-4)(m+1) = 0 \Rightarrow m = 4, -1$

$$\therefore \quad y = c_1 e^{4t} + c_2 e^{-t} \quad \ldots (3)$$

By considering $\frac{dy}{dt} = y - 2x$ we get $x = \frac{1}{2}\left\{ y - \frac{dy}{dt} \right\}$

$$x = \frac{1}{2}\left\{c_1 e^{4t} + c_2 e^{-t} - (4c_1 e^{4t} - c_2 e^{-t})\right\}$$

$$x = \frac{1}{2}(-3c_1 e^{4t} + 2c_2 e^{-t}) \quad \ldots (4)$$

We have conditions $x = 8$, $y = 3$ at $t = 0$

Hence (3) and (4) becomes $c_1 + c_2 = 3$ and $-3c_1/2 + c_2 = 8$

Solving these equations we get $c_2 = 5$, $c_1 = -2$

Thus $x = 3e^{4t} + 5e^{-t}$, $y = -2e^{4t} + 5e^{-t}$ is the required solution.

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75. *Solve :* $\frac{dx}{dt} - 2y = \cos 2t$, $\frac{dy}{dt} + 2x = \sin 2t$ *given that* $x = 1$, $y = 0$ *at* $t = 0$

>> Denoting $D = \frac{d}{dt}$ we have the system of equations

$$Dx - 2y = \cos 2t \quad \ldots (1)$$
$$2x + Dy = \sin 2t \quad \ldots (2)$$

Operating (1) by D and multiplying (2) by 2 we have

$$D^2 x - 2Dy = D(\cos 2t) = -2 \sin 2t$$

$$4x + 2Dy = 2 \sin 2t$$

Adding we get, $(D^2 + 4)x = 0$

A.E is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$$\therefore \quad x = c_1 \cos 2t + c_2 \sin 2t \quad \ldots (3)$$

By considering $\frac{dx}{dt} - 2y = \cos 2t$ we get, $y = \frac{1}{2}\left[\frac{dx}{dt} - \cos 2t\right]$

$$y = \frac{1}{2}\left[\frac{d}{dt}(c_1 \cos 2t + c_2 \sin 2t) - \cos 2t\right]$$

$$= \frac{1}{2}[-2c_1 \sin 2t + 2c_2 \cos 2t - \cos 2t]$$

$$\therefore \quad y = -c_1 \sin 2t + (c_2 - 1/2) \cos 2t \quad \ldots (4)$$

(3) and (4) represents the general solution of the given system of equations.

We shall obtain the particular solution by applying the given conditions.

$x = 1$ at $t = 0$; Hence (3) becomes $1 = c_1 + 0 \quad \therefore \; c_1 = 1$

$y = 0$ at $t = 0$; Hence (4) becomes $0 = 0 + (c_2 - 1/2) \; \therefore \; c_2 = 1/2$

Thus by substituting these values in (3) and (4) we get

$$x = \cos 2t + (\sin 2t/2); \quad y = -\sin 2t.$$

2.7 Applications

76. *A particle moves along the x-axis according to the law* $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0.$ *If the particle is started at* $x = 0$ *with an initial velocity of 12 ft/sec to the left, determine* x *in terms of* t.

>> We have, $(D^2 + 6D + 25)\,x(t) = 0$ and $x = 0$ at $t = 0$, $\frac{dx}{dt} = -12$ at $t = 0$.

(Negative sign is due to the movement along the x-axis to the left)

A.E is $m^2 + 6m + 25 = 0$ and by solving

$$m = \frac{-6 \pm \sqrt{36 - 100}}{2} = \frac{-6 \pm \sqrt{64\,i^2}}{2} = \frac{-6 \pm 8\,i}{2} = -3 \pm 4\,i$$

$\therefore \quad x = x(t) = e^{-3t}\,(c_1 \cos 4t + c_2 \sin 4t) \qquad \ldots (1)$

Now, $\frac{dx}{dt} = e^{-3t}\,(-4c_1 \sin 4t + 4c_2 \cos 4t) - 3e^{-3t}\,(c_1 \cos 4t + c_2 \sin 4t) \qquad \ldots (2)$

Using the initial conditions (1) and (2) respectively becomes,

$0 = c_1$ and $-12 = 4c_2 - 3c_1$

$\therefore \quad c_1 = 0$ and $c_2 = -3$

Thus $\mathbf{x(t) = -3\,e^{-3t} \sin 4t}$

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77. *A particle undergoes forced vibrations according to the law* $x''(t) + 25x(t) = 21 \cos 2t$. *If the particle starts from rest at* $t = 0$, *find the dispacement at any time* $t > 0$.

>> We have $(D^2 + 25)\,x(t) = 21 \cos 2t$ and $x(0) = 0$, $x'(0) = 0$.

A.E is $m^2 + 25 = 0 \quad \therefore \quad m = \pm 5i$ and $x_c = c_1 \cos 5t + c_2 \sin 5t$

$$x_p = \frac{21 \cos 2t}{D^2 + 25}\ ; \quad D^2 \to -4 \text{ and hence } x_p = \frac{21 \cos 2t}{21} = \cos 2t$$

$$x = x(t) = x_c + x_p$$

Hence $x(t) = c_1 \cos 5t + c_2 \sin 5t + \cos 2t \qquad \ldots (1)$

Now, $x'(t) = -5c_1 \sin 5t + 5c_2 \cos 5t - 2 \sin 2t \qquad \ldots (2)$

Using the initial conditions (1) and (2) respectively becomes,

$$0 = c_1 + 1, \quad 0 = 5c_2$$

$\therefore \quad c_1 = -1$ and $c_2 = 0$

Thus $\quad x(t) = \cos 2t - \cos 5t$

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78. *Solve the problem of undamped forced vibrations of a spring governed by the d.e.* $m\dfrac{d^2y}{dt^2} + ky = f(t)$, *in the case where the forcing function is* $f(t) = A\sin wt$. *Take* $y(0) = y_0$ *and* $y'(0) = y_1$

>> We have, $\dfrac{d^2y}{dt^2} + \dfrac{k}{m}y = \dfrac{A}{m}\sin wt$

Let $\quad \lambda^2 = k/m$ and $\mu = A/m$, for convenience.

Now we have, $(D^2 + \lambda^2)\, y(t) = \mu \sin wt$

A.E is $m^2 + \lambda^2 = 0 \quad \therefore \quad m = \pm i\lambda$ and $y_c = c_1 \cos\lambda t + c_2 \sin\lambda t$

$$y_p = \frac{\mu \sin wt}{D^2 + \lambda^2} \ ; \ D^2 \to -w^2 \text{ gives } y_p = \frac{\mu \sin wt}{\lambda^2 - w^2}, \quad (\lambda \neq w)$$

$$y = y_c + y_p$$

Hence, $y(t) = c_1 \cos\lambda t + c_2 \sin\lambda t + \dfrac{\mu \sin wt}{\lambda^2 - w^2}$ ...(1)

Also, $\quad y'(t) = -\lambda c_1 \sin\lambda t + \lambda c_2 \cos\lambda t + \dfrac{\mu w \cos wt}{\lambda^2 - w^2}$ ...(2)

Consider $y(0) = y_0$ and $y'(0) = y_1$

(1) and (2) respectively becomes,

$$y_0 = c_1 \text{ and } y_1 = \lambda c_2 + \frac{\mu w}{\lambda^2 - w^2}$$

$$\therefore \quad c_1 = y_0 \text{ and } c_2 = \frac{1}{\lambda}\left(y_1 - \frac{\mu w}{\lambda^2 - w^2}\right)$$

Using these values in (1) we have,

$$y(t) = y_0 \cos \lambda t + \frac{1}{\lambda}\left(y_1 - \frac{\mu w}{\lambda^2 - w^2}\right) \sin \lambda t + \frac{\mu \sin wt}{\lambda^2 - w^2}$$

Thus $y(t) = y_0 \cos \lambda t + \frac{y_1}{\lambda} \sin\lambda t + \frac{\mu}{\lambda^2 - w^2}\left[\sin wt - \frac{w}{\lambda} \sin\lambda t\right], \lambda \neq w$

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79. *The d.e of a simple pendulum is* $\frac{d^2 x}{dt^2} + w^2 x = F \sin nt$, *where* w *and* F *are constants. If at* $t = 0$, $x = 0$ *and* $\frac{dx}{dt} = 0$, *determine the motion when* $n = w$.

>> We have $(D^2 + w^2)\, x = F \sin nt$

A.E is $m^2 + w^2 = 0 \quad \therefore \quad m = \pm iw$ and $x_c = c_1 \cos wt + c_2 \sin wt$

$$x_p = \frac{F \sin nt}{D^2 + w^2}$$

If $n = w$, $x_p = \frac{F \sin wt}{D^2 + w^2}$; $D^2 \to -w^2$ makes the denominatior zero.

$$x_p = t \cdot \frac{F \sin wt}{2D} = \frac{Ft}{2} \int \sin wt \, dt = \frac{-Ft}{2w} \cos wt$$

$$x = x_c + x_p$$

Hence $x(t) = c_1 \cos wt + c_2 \sin wt - \frac{Ft}{2w} \cos wt$ ... (1)

Also, $x'(t) = -w c_1 \sin wt + w c_2 \cos wt - \frac{F}{2w}(-tw \sin wt + \cos wt)$ ... (2)

We have $x(0) = 0$ and $x'(0) = 0$, by data.

Hence (1) and (2) respectively becomes,

$$0 = c_1 \text{ and } 0 = w c_2 - \frac{F}{2w} \quad \therefore \quad c_1 = 0 \text{ and } c_2 = \frac{F}{2w^2}$$

Thus the motion when $n = w$ is, $x(t) = \frac{F}{2w^2} \sin wt - \frac{Ft}{2w} \cos wt$

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80. *The current i and the charge q in a series circuit containing an inductance L, capacitance C, e.m.f E satisfy the d.e.* $L\frac{di}{dt}+\frac{q}{c}\ E;\ i=\frac{dq}{dt}$. *Express q and i in terms of t, given that L, C, E are cosntants and the value of i, q are both zero initially.*

>> Using $i=\frac{dq}{dt}$ in the given d.e. we have,

$$L\frac{d^2q}{dt^2}+\frac{q}{c}=E \quad \text{or} \quad \frac{d^2q}{dt^2}+\frac{q}{LC}=\frac{E}{L}$$

Denoting $\lambda^2=1/LC$ and $\mu=E/L$ we have $(D^2+\lambda^2)\,q=\mu$

A.E is $m^2+\lambda^2=0 \quad \therefore \quad m=\pm i\lambda$

$$C.F=q_c=c_1\cos\lambda t+c_2\sin\lambda t$$

$$P.I=q_p=\frac{\mu}{D^2+\lambda^2}=\frac{\mu e^{0t}}{D^2+\lambda^2}=\frac{\mu}{\lambda^2}$$

$$q=q_c+q_p$$

$$\therefore \quad q(t)=c_1\cos\lambda t+c_2\sin\lambda t+\frac{\mu}{\lambda^2} \qquad \ldots (1)$$

Also $\quad q'(t)=-\lambda c_1\sin\lambda t+\lambda c_2\cos\lambda t \qquad \ldots (2)$

But $\quad q(0)=0$ and $q'(0)=0$ by data.

Hence (1) and (2) respectively becomes,

$$0=c_1+\frac{\mu}{\lambda^2} \quad \text{and} \quad 0=\lambda c_2 \quad \therefore \quad c_1=-\mu/\lambda^2 \quad \text{and} \quad c_2=0$$

Using these values in (1) we have,

$$q(t)=-(\mu/\lambda^2)\cos\lambda t+(\mu/\lambda^2)$$

or $\quad q(t)=(\mu/\lambda^2)\,[1-\cos\lambda t]$ where $\mu/\lambda^2=\frac{E/L}{1/LC}=EC$

Thus $\quad \boldsymbol{q(t)=EC[1-\cos\sqrt{1/LC}\ t]}$

Also $\quad \boldsymbol{i(t)=q'(t)=E\sqrt{C/L}\ \sin\sqrt{1/LC}\ t}$

EXERCISES

Solve the following differential equations

1. $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = 5e^{2x} + 6e^{-x} + 7$

2. $y'' - 6y' + 13y = e^{2x} + 2^x$

3. $4x''(t) - x(t) = e^{t/2} + 12\cosh t$

4. $y'' - 3y' + 2y = 2\sin x\cos x$

5. $y''' - 3y'' + 9y' - 27y = \cos 3x$

6. $x''(t) + 8x'(t) + 25x(t) = 16(3\cos t - \sin t)$

7. $y'' + y' + y = x^2 + x + 1$

8. $\dfrac{d^3y}{dx^3} - 8y = x(x^2 + 1)$

9. $x'''(t) - x''(t) - 6x'(t) = 1 + t^2$

10. $\dfrac{d^2y}{dx^2} + 4y = 2e^{2x}\sin^2 x$

11. $(D^3 - D^2 + 4D - 4)y = 68e^x\sin 2x$

12. $y'' + 4y' + 3y = e^{-3x}x^2$

13. $x''(t) + 4x'(t) = 65\cosh 2t\cos t$

14. $y'' + 2y' + y = x\sin x$

15. $y'' + y = x^2\sin 2x$

16. $y''' - 3y'' + 4y' - 2y = e^x + \sin(\pi/2 + x)$

17. $\dfrac{d^4y}{dx^4} - y = \cosh(x + 2) + 4\sin(x/2)\cos(x/2)$

18. $(D^4 + 2D^2 + 1)y = x^2\cos x$

19. $\dfrac{dx}{dt} + y = e^t,\ \dfrac{dy}{dt} - x = e^{-t}$

20. $\dfrac{dx}{dt} + x - y = e^t,\ \dfrac{dy}{dt} + y - x = 0$

ANSWERS

1. $y = (c_1 + c_2 x + c_3 x^2) e^{-x} + 5e^{2x}/27 + x^3 e^{-x} + 7$

2. $y = e^{3x} (c_1 \cos 2x + c_2 \sin 2x) + \dfrac{e^{2x}}{5} + \dfrac{2^x}{(\log 2)^2 - 6(\log 2) + 13}$

3. $x = c_1 e^{t/2} + c_2 e^{-t/2} + (t/2) e^{t/2} + 4 \cosh t.$

4. $y = c_1 e^{x} + c_2 e^{2x} + (3 \cos 2x - \sin 2x)/20$

5. $y = c_1 e^{3x} + c_2 \cos 3x + c_3 \sin 3x - x(\sin 3x + \cos 3x) / 36$

6. $x = e^{-4t} (c_1 \cos 3t + c_2 \sin 3t) + 2 \cos t$

7. $y = e^{-x/2} \{c_1 \cos(\sqrt{3}\, x/2) + c_2 \sin(\sqrt{3}\, x/2)\} + x^2 - x$

8. $y = c_1 e^{2x} + e^{-x} \{c_2 \cos(\sqrt{3}\, x) + c_3 \sin(\sqrt{3}\, x)\} - \dfrac{(4x^3 + 4x + 3)}{32}$

9. $x = c_1 + c_2 e^{3t} + c_3 e^{-2t} - \dfrac{t}{108}(6t^2 - 3t + 25)$

10. $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{e^{2x}}{8} - \dfrac{e^{2x}}{20}(2 \sin 2x + \cos 2x)$

11. $y = c_1 e^{x} + c_2 \cos 2x + c_3 \sin 2x - 2e^{x}(4 \sin 2x + \cos 2x)$

12. $y = c_1 e^{-x} + c_2 e^{-3x} - \dfrac{e^{-3x}}{24}(4x^3 + 6x^2 + 6x + 3)$

13. $x = c_1 \cos 2t + c_2 \sin 2t + 4 \sinh 2t \sin t + 7 \cosh 2t \cos t$

14. $y = (c_1 + c_2 x) e^{-x} + \dfrac{1}{2}(\sin x - x \cos x + \cos x)$

15. $y = c_1 \cos x + c_2 \sin x + \dfrac{1}{27}(26 \sin 2x - 24 x \cos 2x - 9x^2 \sin 2x)$

16. $y = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x) + x e^x + \frac{1}{10} (\cos x + 3 \sin x)$

17. $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \frac{x \sinh (x + 2)}{4} + \frac{x \cos x}{2}$

18. $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x + \frac{x^3 \sin x}{12} + \frac{9x^2 - x^4}{48} \cos x$

19. $x = - c_1 \sin t + c_2 \cos t + \sinh t , \; y = c_1 \cos t + c_2 \sin t + \sinh t$

20. $x = c_1 + c_2 e^{-2t} + 2 e^t/3 \; ; \; y = c_1 - c_2 e^{-2t} + e^t/3$

PART A

Unit - III

Differential Equation - 3

3.1 Introduction

In this unit, we first discuss a special method for solving a nonhomogeneous linear differential equation of second order involved with complicated functions on the right hand side of the equation where the particular integral cannot be found by inverse differential operator method. The method can be extended for higer order equations also.

We also discuss solution of differential equations in some specific forms involving independent variable as coefficients of the derivatives. (*D.Es with variable coefficients*
In fact these equations are solved by reducing into equations with constant coefficients

Further we also discuss series solution for a homogeneous differential equation of second order involved with constant/variable coefficients. The solution will contain infinite series.

3.2 Method of variation of parameters

Consider a second order D.E in the form

$$y'' + a_1 y' + a_2 y = \phi(x) \qquad \ldots (1)$$

where a_1, a_2 may be functions of x or constants. Let us suppose that the C.F associated with (1) is in the form $y_c = c_1 y_1 + c_2 y_2$ where c_1, c_2 are arbitrary constants and y_1, y_2 are functions of x being the linearly independent solutions of the homogeneous equation $y'' + a_1 y' + a_2 y = 0$.

This implies that

$$y_1'' + a_1 y_1' + a_2 y_1 = 0 \qquad \ldots (2)$$

$$y_2'' + a_2 y_2' + a_2 y_2 = 0 \qquad \ldots (3)$$

We replace the arbitrary constants c_1, c_2 present in y_c by functions of x say A, B respectively and try to determine them such that

$$y = A y_1 + B y_2 \quad \ldots (4)$$

is the complete / general solution of the given equation.

The procedure to determine $A(x)$ and $B(x)$ is as follows.

From (4) $y' = (A y_1' + B y_2') + (A' y_1 + B' y_2)$ $\ldots (5)$

We shall choose A and B such that

$$A' y_1 + B' y_2 = 0 \quad \ldots (6)$$

Thus (5) becomes $y' = A y_1' + B y_2'$ $\ldots (7)$

Differentiating (7) w.r.t x again we have,

$$y'' = (A y_1'' + B y_2'') + (A' y_1' + B' y_2') \quad \ldots (8)$$

Thus (1) as a consequence of (4),(7) and (8) becomes

$$(A y_1'' + B y_2'' + A' y_1' + B' y_2') + a_1 (A y_1' + B y_2') + a_2 (A y_1 + B y_2) = \phi(x)$$

ie., $A(y_1'' + a_1 y_1' + a_2 y_1) + B(y_2'' + a_1 y_2' + a_2 y_2) + (A' y_1' + B' y_2') = \phi(x)$

ie., $A \cdot 0 + B \cdot 0 + (A' y_1' + B' y_2') = \phi(x)$, by using (2) and (3)

ie., $A' y_1' + B' y_2' = \phi(x)$ $\ldots (9)$

Let us consider equations (6) and (9) for solving.

$$A' y_1 + B' y_2 = 0 \quad \ldots (6)$$

$$A' y_1' + B' y_2' = \phi(x) \quad \ldots (9)$$

Solving for A', B' by Cramer's rule we have,

$$\frac{A'}{\begin{vmatrix} 0 & y_2 \\ \phi(x) & y_2' \end{vmatrix}} = \frac{B'}{\begin{vmatrix} y_1 & 0 \\ y_1' & \phi(x) \end{vmatrix}} = \frac{1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

ie., $$\frac{A'}{-y_2 \phi(x)} = \frac{B'}{y_1 \phi(x)} = \frac{1}{y_1 y_2' - y_2 y_1'} = \frac{1}{W} (say)$$

where $W = y_1 y_2' - y_2 y_1'$ is called the *Wronskian* of the functions y_1, y_2 It should be noted that $W \neq 0$ if y_1 and y_2 are independent.

Hence $\dfrac{A'}{-y_2 \phi(x)} = \dfrac{B'}{y_1 \phi(x)} = \dfrac{1}{W}$ will give us

$$A' = \frac{-y_2 \phi(x)}{W}, \quad B' = \frac{y_1 \phi(x)}{W}$$

$\Rightarrow \quad A = -\int \frac{y_2 \phi(x)}{W} dx + k_1, \; B = \int \frac{y_1 \phi(x)}{W} dx + k_2$

Substituting the expressions of $A(x)$, $B(x)$ obtained from these in $y = A y_1 + B y_2$ we obtain the complete solution of the given differential equation.

Note : **1.** *In a similar way the method can be extended for higher order equations also.*

2. *Since the constants c_1 and c_2 present in y_c are replaced by functions of x, the method is called the method of variation of parameters.*

3. *The inverse differential operator method can be applied only to the equations having some specific form of functions in the R.H.S. But the method of variation of parameters donot have such restrictions and hence it is a powerful method for solving a higher order non homogeneous equation.*

Working procedure for problems *(In the case of a second order equation)*

- Given $f(D)y = \phi(x)$ we write C.F in the form $y_c = c_1 y_1 + c_2 y_2$
- We assume $y = A y_1 + B y_2$ to be the complete solution of the D.E where A, B are functions of x.
- We compute y_1', y_2' and $W = y_1 y_2' - y_2 y_1'$
- We assume the expressions for A' and B' in the form :

$$A' = \frac{-y_2 \phi(x)}{W}, \; B' = \frac{y_1 \phi(x)}{W}$$

and simplify for the purpose of integration.

- $A = -\int \frac{y_2 \phi(x)}{W} dx + k_1, \; B = \int \frac{y_1 \phi(x)}{W} dx + k_2$
- $A(x)$, $B(x)$ are substituted in $y = A y_1 + B y_2$

WORKED PROBLEMS

1. *Solve by the method of variation of parameters.* $y'' + a^2 y = \sec ax$

>> We have $(D^2 + a^2)y = \sec ax$

A.E is $m^2 + a^2 = 0 \Rightarrow m = \pm ai$

$\therefore \quad y_c = c_1 \cos ax + c_2 \sin ax$

$y = A(x)\cos ax + B(x)\sin ax \quad \ldots (1)$

be the complete solution of the given equation where $A(x)$, $B(x)$ are to be found.

We have $y_1 = \cos ax \qquad y_2 = \sin ax$

$$y_1' = -a\sin ax \quad y_2' = a\cos ax$$

$$W = y_1 y_2' - y_2 y_1' = a. \quad \text{Also } \phi(x) = \sec ax$$

$$A' = \frac{-y_2\,\phi(x)}{W} \qquad B' = \frac{y_1\,\phi(x)}{W}$$

$$A' = \frac{-\sin ax \sec ax}{a} \qquad B' = \frac{\cos ax \cdot \sec ax}{a}$$

ie., $A' = \dfrac{-\tan ax}{a} \qquad B' = \dfrac{1}{a}$

$\Rightarrow \quad A = \dfrac{-1}{a}\displaystyle\int \tan ax\,dx + k_1 \qquad B = \dfrac{1}{a}\displaystyle\int dx + k_2$

ie., $A = \dfrac{-\log(\sec ax)}{a^2} + k_1 \qquad B = \dfrac{x}{a} + k_2$

Substituting these in (1) we have,

$$y = \left[\frac{-\log(\sec ax)}{a^2} + k_1\right]\cos ax + \left[\frac{x}{a} + k_2\right]\sin ax$$

Thus $\boldsymbol{y = k_1 \cos ax + k_2 \sin ax - \dfrac{\cos ax \log(\sec ax)}{a^2} + \dfrac{x \sin ax}{a}}$

---

2. *Solve :* $\dfrac{d^2y}{dx^2} + y = \tan x$ *by the method of variation of parameters.*

>> We have $(D^2 + 1)\,y = \tan x.$

A.E is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore \quad y_c = c_1 \cos x + c_2 \sin x.$

$$y = A(x)\cos x + B(x)\sin x \qquad \ldots (1)$$

be the complete solution of the given D.E where $A(x)$, $B(x)$ are to be found.

We have $y_1 = \cos x \qquad y_2 = \sin x$

$$y_1' = -\sin x \qquad y_2' = \cos x$$

$$W = y_1 y_2' - y_2 y_1' = 1. \quad \text{Also } \phi(x) = \tan x$$

$$A' = \frac{-y_2\,\phi(x)}{W} \qquad B' = \frac{y_1\,\phi(x)}{W}$$

$A' = -\sin x \tan x$ $\qquad B' = \cos x \tan x$

$A' = \dfrac{-\sin^2 x}{\cos x}$ $\qquad B' = \sin x$

$\Rightarrow \quad A = \int \dfrac{-(1-\cos^2 x)}{\cos x} dx + k_1$ $\qquad B = \int \sin x\, dx + k_2$

$A = \int (\cos x - \sec x)\, dx + k_1$ $\qquad B = -\cos x + k_2$

$A = \sin x - \log(\sec x + \tan x) + k_1$ $\qquad B = -\cos x + k_2$

Substituting these in (1) we have,

$$y = \left\{\sin x - \log(\sec x + \tan x) + k_1\right\}\cos x + \left\{-\cos x + k_2\right\}\sin x$$

Thus $\quad \boldsymbol{y = k_1 \cos x + k_2 \sin x - \cos x \log(\sec x + \tan x)}$

---

3. *Solve* $\dfrac{d^2 y}{dx^2} + y = \sec x \tan x$ *by the method of variation of parameters.*

Note : *The problem is similar to the previous one. But we have* $\phi(x) = \sec x \tan x$. *The computation of A and B is given.*

$A' = -\sin x \sec x \tan x$, $\qquad B' = \cos x \sec x \tan x$

$A' = -\tan^2 x = 1 - \sec^2 x$ $\qquad B' = \tan x$

$\Rightarrow \quad A = \int (1 - \sec^2 x)\, dx + k_1$ $\qquad B = \int \tan x\, dx + k_2$

$A = x - \tan x + k_1$ $\qquad B = \log(\sec x) + k_2$

Substituting these in $y = A\cos x + B\sin x$ we have,

$$y = \left\{x - \tan x + k_1\right\}\cos x + \left\{\log(\sec x) + k_2\right\}\sin x$$

$$y = k_1 \cos x + k_2 \sin x + x\cos x - \sin x + \sin x \log(\sec x)$$

The term $-\sin x$ can be neglected in view of the term $k_2 \sin x$ present in the solution.

Thus $\quad \boldsymbol{y = k_1 \cos x + k_2 \sin x + x\cos x + \sin x \log(\sec x)}$

---

4. *Solve* $(D^2 + 1)y = \operatorname{cosec} x \cot x$ *by the method of variation of parmeters.*

Note : *This problem is also similar to the earlier two problems. But we have* $\phi(x) = \operatorname{cosec} x \cot x$. *The computation of A and B is given.*

$A' = -\sin x \operatorname{cosec} x \cot x$ $\qquad B' = \cos x \operatorname{cosec} x \cot x$

$$A' = -\cot x \qquad B' = \cot^2 x$$

$$\Rightarrow \quad A = \int -\cot x\, dx + k_1 \qquad B = \int (\operatorname{cosec}^2 x - 1)\, dx + k_2$$

$$A = -\log(\sin x) + k_1 \qquad B = -\cot x - x + k_2$$

Substituting these in $y = A\cos x + B\sin x$ we have,

$$y = \{-\log(\sin x) + k_1\}\cos x + \{-\cot x - x + k_2\}\sin x$$

$$y = k_1\cos x + k_2\sin x - \cos x\log(\sin x) - \cos x - x\sin x.$$

The term $-\cos x$ can be neglected in view of the term $k_1\cos x$ present in the solution.

Thus $\boldsymbol{y = k_1\cos x + k_2\sin x - \cos x\log(\sin x) - x\sin x}$

---

5. *Solve by the method of variation of parmeters.* $y'' + 4y = 4\sec^2 2x$

>> We have $(D^2 + 4)y = 4\sec^2 2x$

A.E is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$$\therefore \quad y_c = c_1\cos 2x + c_2\sin 2x.$$

$$y = A(x)\cos 2x + B(x)\sin 2x \qquad \ldots (1)$$

be the complete solution of the given equation where $A(x)$, $B(x)$ are to be found.

We have $\quad y_1 = \cos 2x \qquad y_2 = \sin 2x$

$$y_1' = -2\sin 2x \qquad y_2' = 2\cos 2x$$

$$W = y_1 y_2' - y_2 y_1' = 2. \quad \text{Also } \phi(x) = 4\sec^2 2x$$

$$A' = \frac{-y_2\,\phi(x)}{W} \qquad B' = \frac{y_1\,\phi(x)}{W}$$

$$A' = \frac{-\sin 2x \cdot 4\sec^2 2x}{2} \qquad B' = \frac{\cos 2x \cdot 4\sec^2 2x}{2}$$

$$A' = -2\tan 2x\sec 2x \qquad B' = 2\sec 2x$$

$$\Rightarrow \quad A = \int -2\tan 2x\sec 2x\, dx + k_1\ ; \qquad B = \int 2\sec 2x\, dx + k_2$$

$$A = -\sec 2x + k_1 \qquad B = \log(\sec 2x + \tan 2x) + k_2$$

Substituting these in (1) we have,

$$y = \left\{ -\sec 2x + k_1 \right\} \cos 2x + \left\{ \log(\sec 2x + \tan 2x) + k_2 \right\} \sin 2x$$

Thus $\boldsymbol{y = k_1 \cos 2x + k_2 \sin 2x - 1 + \sin 2x \cdot \log(\sec 2x + \tan 2x)}$

---

6. *Using the method of variation of parameters solve* $\dfrac{d^2 y}{dx^2} + 4y = \tan 2x$

Note : *This problem is similar to the previous one. But we have* $\phi(x) = \tan 2x$. *The computation of A and B is given.*

>> Now $A' = \dfrac{-y_2 \phi(x)}{W}$; $B' = \dfrac{y_1 \phi(x)}{W}$

ie., $A' = \dfrac{-\sin 2x \tan 2x}{2}$; $B' = \dfrac{\cos 2x \tan 2x}{2}$

Consider $A' = \dfrac{-\sin^2 2x}{2\cos 2x} = \dfrac{\cos^2 2x - 1}{2\cos 2x}$ $\qquad B' = \dfrac{\sin 2x}{2}$

$$A = \int \left[\frac{\cos 2x}{2} - \frac{\sec 2x}{2} \right] dx + k_1 \qquad B = \int \frac{\sin 2x}{2}\, dx + k_2$$

$$A = \frac{\sin 2x}{4} - \frac{\log(\sec 2x + \tan 2x)}{4} + k_1 \qquad B = -\frac{\cos 2x}{4} + k_2$$

Using the expressions of A and B in $y = A \cos 2x + B \sin 2x$ we have,

$$y = \left[\frac{\sin 2x}{4} - \frac{\log(\sec 2x + \tan 2x)}{4} + k_1 \right] \cos 2x + \left[\frac{-\cos 2x}{4} + k_2 \right] \sin 2x$$

ie.,

$$y = k_1 \cos 2x + k_2 \sin 2x + \frac{\sin 2x \cos 2x}{4} - \frac{\cos 2x \log(\sec 2x + \tan 2x)}{4} - \frac{\sin 2x \cos 2x}{4}$$

Thus $\boldsymbol{y = k_1 \cos 2x + k_2 \sin 2x - \dfrac{\cos 2x \log(\sec 2x + \tan 2x)}{4}}$

---

7. *Using the method of variation of parameters solve* : $\dfrac{d^2 y}{dx^2} + y = \dfrac{1}{1 + \sin x}$

>> We have $(D^2 + 1)\, y = \dfrac{1}{1 + \sin x}$

A.E. is given by $m^2 + 1 = 0$ and hence $m = \pm i$

$\therefore \quad y_c = c_1 \cos x + c_2 \sin x$

$$y = A(x)\cos x + B(x)\sin x \qquad \ldots(1)$$

be the complete solution of the given d.e where $A(x)$ and $B(x)$ are to be found.

We have $y_1 = \cos x \quad ; \quad y_2 = \sin x$

$$y_1' = -\sin x \quad ; \quad y_2' = \cos x$$

$$W = y_1 y_2' - y_2 y_1' = 1. \quad \text{Also } \phi(x) = \frac{1}{1+\sin x}$$

Now $\quad A' = \dfrac{-y_2\,\phi(x)}{W}$ and $B' = \dfrac{y_1\,\phi(x)}{W}$

ie., $\quad A' = \dfrac{-\sin x}{1+\sin x}$ and $B' = \dfrac{\cos x}{1+\sin x}$

Consider $A' = \dfrac{-(1+\sin x - 1)}{1+\sin x} = -1 + \dfrac{1}{1+\sin x}$

$$\Rightarrow \quad A = \int\left[-1 + \frac{1}{1+\sin x}\right]dx + k_1$$

$$= -x + \int \frac{1-\sin x}{\cos^2 x}\,dx + k_1$$

$$= -x + \int(\sec^2 x - \sec x \tan x)\,dx + k_1$$

$$A = -x + \tan x - \sec x + k_1 \qquad \ldots(2)$$

Also $\quad B' = \dfrac{\cos x}{1+\sin x} = \dfrac{\cos x(1-\sin x)}{\cos^2 x} = \dfrac{1-\sin x}{\cos x}$

$$\Rightarrow \quad B = \int \frac{1-\sin x}{\cos x}\,dx + k_2$$

$$= \int(\sec x - \tan x)\,dx + k_2$$

$$= \log(\sec x + \tan x) + \log(\cos x) + k_2$$

$$= \log\left(\frac{1+\sin x}{\cos x}\right) + \log(\cos x) + k_2$$

$$= \log(1+\sin x) - \log(\cos x) + \log(\cos x) + k_2$$

$$B = \log(1+\sin x) + k_2 \qquad \ldots(3)$$

Using (2) and (3) in (1) we have,

$$y = [-x + \tan x - \sec x + k_1] \cos x + [\log(1 + \sin x) + k_2] \sin x$$

ie., $y = k_1 \cos x + k_2 \sin x - x \cos x + \sin x - 1 + \sin x \log(1 + \sin x)$

The term $\sin x$ can be neglected in view of the term $k_2 \sin x$ present in the solution.

Thus $\boldsymbol{y = k_1 \cos x + k_2 \sin x - (x \cos x + 1) + \sin x \log(1 + \sin x)}$

---

8. *Solve :* $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + 2y = e^x \tan x$ *using the method of variation of parameters.*

>> We have $(D^2 - 2D + 2) y = e^x \tan x$; $D = \dfrac{d}{dx}$

A.E. is given by $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$\therefore$ $y_c = e^x (c_1 \cos x + c_2 \sin x)$

Let $y = e^x (A \cos x + B \sin x)$ be the general solution of the given equation where A and B are functions of x to be found.

We have $y_1 = e^x \cos x$; $y_2 = e^x \sin x$

$\therefore$ $y_1' = e^x (\cos x - \sin x)$; $y_2' = e^x (\cos x + \sin x)$

$$W = y_1 y_2' - y_2 y_1' = e^{2x} (\cos^2 x + \sin^2 x) = e^{2x} ; \phi(x) = e^x \tan x$$

$$A' = -\frac{y_2 \phi(x)}{W} \quad ; \quad B' = \frac{y_1 \phi(x)}{W}$$

$$A' = -\frac{e^{2x} \sin x \tan x}{e^{2x}} \quad ; \quad B' = \frac{e^{2x} \cos x \tan x}{e^{2x}}$$

$$A' = -\frac{\sin^2 x}{\cos x} = -\frac{(1 - \cos^2 x)}{\cos x} \quad ; \quad B' = \sin x$$

$\Rightarrow$ $A = \int (\cos x - \sec x)\, dx + k_1$; $B = \int \sin x \, dx + k_2$

$$A = \sin x - \log(\sec x + \tan x) + k_1 ; B = -\cos x + k_2$$

Substituting these in $y = e^x (A \cos x + B \sin x)$ we get,

$$y = e^x \left\{ \sin x - \log(\sec x + \tan x) + k_1 \right\} \cos x + e^x \left\{ -\cos x + k_2 \right\} \sin x$$

Thus $\boldsymbol{y = e^x (k_1 \cos x + k_2 \sin x) - e^x \log(\sec x + \tan x) \cos x}$

---

9. *Using the method of variation of parameters find the solution of* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}$

>> We have, $(D^2 - 2D + 1)\,y = \frac{e^x}{x}$, where $D = \frac{d}{dx}$

A.E is given by $m^2 - 2m + 1 = 0$ or $(m-1)^2 = 0$

$m = 1, 1$ are the roots of A.E

$\therefore \quad y_c = (c_1 + c_2 x)\,e^x$

$y = (A + Bx)\,e^x$ where $A = A(x), B = B(x)$ be the complete solution of the d.e and we shall find A, B.

We have, $y_1 = e^x \;;\; y_2 = x\,e^x$

$\therefore \quad y_1' = e^x \;;\; y_2' = (x+1)\,e^x$

$W = y_1 y_2' - y_2 y_1' = e^{2x}$. Also $\phi(x) = e^x/x$

Further we have, $A' = \frac{-y_2\,\phi(x)}{W}$ and $B' = \frac{y_1\,\phi(x)}{W}$

$A' = \frac{-xe^x \cdot e^x/x}{e^{2x}} \quad ; \quad B' = \frac{e^x \cdot e^x/x}{e^{2x}}$

i.e., $\quad A' = -1 \quad ; \quad B' = 1/x$

$\Rightarrow \quad A = \int -1\,dx + k_1 \quad ; \quad B = \int \frac{1}{x}\,dx + k_2$

i.e., $\quad A = -x + k_1 \quad ; \quad B = \log x + k_2$

Using these in $y = A\,e^x + B\,x\,e^x$ we have,

$$y = (-x + k_1)\,e^x + (\log x + k_2)\,x\,e^x$$

i.e., $\quad y = (k_1 + k_2 x)\,e^x + (\log x - 1)\,xe^x$

The term $-xe^x$ can be neglected in view of the term $k_2\,xe^x$ present in the solution.

Thus $\quad y = (k_1 + k_2 x)\,e^x + x\log x\,e^x$

---

10. *By the method of variation of parameters solve :* $y'' - 2y' + y = e^x \log x$

>> We have $(D^2 - 2D + 1)\, y = e^x \log x$

A.E is $m^2 - 2m + 1 = 0$ or $(m-1)^2 = 0 \Rightarrow m = 1, 1$

$\therefore \quad y_c = (c_1 + c_2 x)\, e^x$

$y = A\,e^x + B\,x\,e^x$ be the complete solution of the given equation where A and B are functions of x to be found.

We have $\quad y_1 = e^x \quad ; \quad y_2 = x\,e^x$

$\quad y_1' = e^x \quad ; \quad y_2' = x\,e^x + e^x$

$W = y_1 y_2' - y_2 y_1' = x\,e^{2x} + e^{2x} - x\,e^{2x} = e^{2x}$. Also $\phi(x) = e^x \log x$

$$A' = \frac{-y_2\,\phi(x)}{W} \qquad B' = \frac{y_1\,\phi(x)}{W}$$

$$A' = \frac{-x\,e^x \cdot e^x \log x}{e^{2x}} \qquad B' = \frac{e^x \cdot e^x \log x}{e^{2x}}$$

$$\Rightarrow \quad A = -\int \log x \cdot x\,dx \qquad B = \int \log x \cdot 1\,dx$$

Integrating both these terms by parts we get,

$$A = -\left[\log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx\right] + k_1 \qquad B = \log x \cdot x - \int x \cdot \frac{1}{x}\,dx + k_2$$

$$A = \frac{-x^2 \log x}{2} + \frac{x^2}{4} + k_1 \qquad B = x \log x - x + k_2$$

Substituting these in $y = A\,e^x + B\,x\,e^x$ we have,

$$y = \left\{\frac{-x^2 \log x}{2} + \frac{x^2}{4} + k_1\right\} e^x + (x \log x - x + k_2)\, x\,e^x$$

$$y = (k_1 + k_2 x)\, e^x - \frac{x^2 \log x\, e^x}{2} + \frac{x^2 e^x}{4} + x^2 \log x\, e^x - x^2 e^x$$

$$y = (k_1 + k_2 x)\, e^x + \frac{x^2 \log x\, e^x}{2} - \frac{3}{4} x^2 e^x$$

Thus $\quad \mathbf{y = (k_1 + k_2 x)\, e^x + \frac{x^2 e^x}{4}(2 \log x - 3)}$

---

11. *Using the method of variation of parameters solve :* $\dfrac{d^2 y}{dx^2} - 6\dfrac{dy}{dx} + 9y = \dfrac{e^{3x}}{x^2}$

>> We have $(D^2 - 6D + 9)\, y = e^{3x}/x^2$

A.E is $m^2 - 6m + 9 = 0$ or $(m-3)^2 = 0 \Rightarrow m = 3, 3$

$\therefore \quad y_c = (c_1 + c_2 x)\, e^{3x}$

$y = A\, e^{3x} + B\, x\, e^{3x}$ be the complete solution of the given equation where A and B are functions of x to be found.

We have $\quad y_1 = e^{3x} \qquad y_2 = x\, e^{3x}$

$y_1' = 3e^{3x} \qquad y_2' = 3\, x\, e^{3x} + e^{3x}$

$W = y_1 y_2' - y_2 y_1' = e^{6x}$. Also $\phi(x) = e^{3x}/x^2$

$A' = \dfrac{-y_2\, \phi(x)}{W} \qquad B' = \dfrac{y_1\, \phi(x)}{W}$

$A' = \dfrac{-xe^{3x} \cdot e^{3x}/x^2}{e^{6x}} \qquad B' = \dfrac{e^{3x} \cdot e^{3x}/x^2}{e^{6x}}$

$A' = -1/x \qquad B' = 1/x^2$

$\Rightarrow \quad A = \int -1/x\; dx + k_1 \qquad B = \int 1/x^2\, dx + k_2$

ie., $\quad A = -\log x + k_1 \qquad B = -1/x + k_2$

Substituting these in $y = A\, e^{3x} + B\, x\, e^{3x}$ we have,

$$y = (-\log x + k_1)\, e^{3x} + (-1/x + k_2)\, x\, e^{3x}$$

$$y = (k_1 + k_2 x)\, e^{3x} - e^{3x} \log x - e^{3x}$$

The term $-e^{3x}$ can be neglected in view of the term $k_1 e^{3x}$ present in the solution.

Thus $\quad \mathbf{y = (k_1 + k_2 x)\, e^{3x} - e^{3x} \log x}$

---

12. *Solve by the method of variation of parameters* $y'' - 3y' + 2y = \dfrac{1}{1 + e^{-x}}$

>> We have $(D^2 - 3D + 2)\, y = 1/1 + e^{-x}$

A.E is $m^2 - 3m + 2 = 0$ or $(m-1)(m-2) = 0 \Rightarrow m = 1, 2$

$\therefore \quad y_c = c_1 e^x + c_2 e^{2x}$

$y = A e^x + B e^{2x}$ be the complete solution of the given equation where A and B are functions of x to be found.

We have $\quad y_1 = e^x \;;\; y_2 = e^{2x}$

$$y_1' = e^x \;;\; y_2' = 2e^{2x}$$

$$W = y_1 y_2' - y_2 y_1' = e^{3x}. \text{ Also } \phi(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}$$

$$A' = \frac{-y_2\,\phi(x)}{W} \qquad B' = \frac{y_1\,\phi(x)}{W}$$

$$A' = \frac{-e^{2x}\cdot e^x}{(e^x+1)\,e^{3x}} \qquad B' = \frac{e^x\cdot e^x}{(e^x+1)\,e^{3x}}$$

$$A' = \frac{-1}{e^x+1} \qquad B' = \frac{1}{e^x(e^x+1)}$$

$$\Rightarrow \quad A = \int \frac{-dx}{e^x+1} + k_1 \qquad B = \int \frac{dx}{e^x(e^x+1)} + k_2$$

For the first term put $e^x = t \quad \therefore\ e^x dx = dt$ or $dx = dt/e^x = dt/t$

Now $A = -\int \dfrac{dt}{t(t+1)}$ But $\dfrac{1}{t(t+1)} = \dfrac{1}{t} - \dfrac{1}{t+1}$

Hence $A = -\int \left(\dfrac{1}{t} - \dfrac{1}{t+1}\right) dt = \log(t+1) - \log t = \log\left(\dfrac{t+1}{t}\right)$

$$\therefore \quad A = \log\left(\frac{e^x+1}{e^x}\right) + k_1$$

Also consider $B = \displaystyle\int \frac{dx}{e^x(e^x+1)}$

Again by putting $e^x = t$ we get $B = \displaystyle\int \frac{dt}{t^2(t+1)}$

But $\dfrac{1}{t^2(t+1)} = \dfrac{-1}{t} + \dfrac{1}{t^2} + \dfrac{1}{t+1}$ by partial fractions.

$$\int \frac{dt}{t^2(t+1)} = -\int \frac{dt}{t} + \int \frac{dt}{t^2} + \int \frac{dt}{t+1}$$

ie., $B = -\log t - \dfrac{1}{t} + \log(t+1)$ or $B = \log\left(\dfrac{t+1}{t}\right) - \dfrac{1}{t}$

ie., $B = \log\left(\frac{e^x+1}{e^x}\right) - \frac{1}{e^x} + k_2$

Substituting A and B in $y = A\,e^x + B\,e^{2x}$ we have,

$$y = \left[\log\left(\frac{e^x+1}{e^x}\right) + k_1\right]e^x + \left[\log\left(\frac{e^x+1}{e^x}\right) - e^{-x} + k_2\right]e^{2x}$$

$$y = k_1\,e^x + k_2\,e^{2x} + \log(1+e^{-x})\,[\,e^x + e^{2x}\,] - e^x$$

The term $-e^x$ can be neglected inview of the term $k_1\,e^x$ present in the solution.

Thus $\boldsymbol{y = k_1\,e^x + k_2\,e^{2x} + \log(1+e^{-x})\,[\,e^x + e^{2x}\,]}$

---

13. *Solve by the method of variation of parameters* $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$

>> We have $(D^2-1)\,y = 2/1+e^x$

A.E is $(m^2-1) = 0$ or $(m-1)(m+1) = 0 \Rightarrow m = 1, -1$

$\therefore \quad y_c = c_1\,e^x + c_2\,e^{-x}$

$y = A\,e^x + B\,e^{-x}$ be the complete solution of the given equation where A and B are functions of x to be found.

We have $y_1 = e^x \quad ; \quad y_2 = e^{-x}$

$y_1{}' = e^x \quad ; \quad y_2{}' = -e^{-x}$

$W = y_1\,y_2{}' - y_2\,y_1{}' = -2$. Also $\phi(x) = 2/1+e^x$

$$A' = \frac{-y_2\,\phi(x)}{W} \qquad B' = \frac{y_1\,\phi(x)}{W}$$

$$A' = \frac{-2e^{-x}}{-2(1+e^x)} \qquad B' = \frac{2e^x}{-2(1+e^x)}$$

$$\Rightarrow \quad A = \int \frac{dx}{e^x(1+e^x)} + k_1 \qquad B = -\int \frac{e^x}{1+e^x}\,dx + k_2$$

Referring to the previous problem we have,

$$A = \log\left(\frac{e^x+1}{e^x}\right) - e^{-x} + k_1 \quad \text{or} \quad A = \log(1+e^{-x}) - e^{-x} + k_1$$

Also $B = -\log(1+e^x) + k_2$

Substituting $A(x)$, $B(x)$ in $y = A e^{x} + B e^{-x}$ we have,

$$y = \left\{ \log(1+e^{-x}) - e^{-x} + k_1 \right\} e^{x} + \left\{ -\log(1+e^{x}) + k_2 \right\} e^{-x}$$

Thus $\quad \mathbf{y = k_1 e^{x} + k_2 e^{-x} - 1 + e^{x} \log(1+e^{-x}) - e^{-x} \log(1+e^{x})}$

or $\quad y = k_1 e^{x} + k_2 e^{-x} - 1 + e^{x} \log\left(\dfrac{e^{x}+1}{e^{x}} \right) - e^{-x} \log(1+e^{x})$

$$= k_1 e^{x} + k_2 e^{-x} - 1 + e^{x} \left\{ \log(e^{x}+1) - \log e^{x} \right\} - e^{-x} \log(1+e^{x})$$

$$\mathbf{y = k_1 e^{x} + k_2 e^{-x} - 1 + (e^{x} - e^{-x}) \log(1+e^{x}) - x e^{x}}$$

---

4. *Solve by the method of variation of parameters* $y'' + 2y' + 2y = e^{-x} \sec^3 x$

>> We have $(D^2 + 2D + 2) y = e^{-x} \sec^3 x$

A.E is $m^2 + 2m + 2 = 0$ and by solving,

$$m = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$\therefore \quad y_c = e^{-x} (c_1 \cos x + c_2 \sin x)$

$$y = A e^{-x} \cos x + B e^{-x} \sin x$$

be the solution of the given equation where A and B are function of x to be found.

We have $\quad y_1 = e^{-x} \cos x \quad ; \quad y_2 = e^{-x} \sin x$

$$y_1' = -e^{-x} (\sin x + \cos x) \quad ; \quad y_2' = e^{-x} (\cos x - \sin x)$$

$$W = y_1 y_2' - y_2 y_1' = e^{-2x}. \quad \text{Also } \phi(x) = e^{-x} \sec^3 x$$

$$A' = \frac{-y_2 \phi(x)}{W} \qquad B' = \frac{y_1 \phi(x)}{W}$$

$$A' = \frac{-e^{-x} \sin x \cdot e^{-x} \sec^3 x}{e^{-2x}} \qquad B' = \frac{e^{-x} \cos x \cdot e^{-x} \sec^3 x}{e^{-2x}}$$

$$A' = -\tan x \sec^2 x \qquad B' = \sec^2 x$$

$\Rightarrow \quad A = -\int \tan x \sec^2 x \, dx + k_1 \qquad B = \int \sec^2 x \, dx + k_2$

$$A = \frac{-\tan^2 x}{2} + k_1 \qquad B = \tan x + k_2$$

Substituting these in $y = A e^{-x} \cos x + B e^{-x} \sin x$ we have,

$$y = \left(\frac{-\tan^2 x}{2} + k_1\right) e^{-x} \cos x + (\tan x + k_2) e^{-x} \sin x$$

$$y = e^{-x}(k_1 \cos x + k_2 \sin x) - \frac{e^{-x} \tan x \sin x}{2} + e^{-x} \tan x \sin x$$

Thus $\quad \boldsymbol{y = e^{-x}(k_1 \cos x + k_2 \sin x) + \frac{e^{-x} \tan x \sin x}{2}}$

---

15. *Solve* $(D^2 - 3D + 2)\, y = \cos(e^{-x})$ *by the method of variation of parameters.*

>> We have $(D^2 - 3D + 2)\, y = \cos(e^{-x})$

A.E is $m^2 - 3m + 2 = 0$ or $(m-1)(m-2) = 0 \Rightarrow m = 1, 2$

$\therefore \quad y_c = c_1 e^x + c_2 e^{2x}$

$y = A e^x + B e^{2x}$ be the complete solution of the given equation where A and B are functions of x to be found.

We have $\quad y_1 = e^x \quad ; \quad y_2 = e^{2x}$

$y_1' = e^x \quad ; \quad y_2' = 2e^{2x}$

$W = y_1 y_2' - y_2 y_1' = e^{3x}$. Also $\phi(x) = \cos(e^{-x})$

$$A' = \frac{-y_2\,\phi(x)}{W} \qquad B' = \frac{y_1\,\phi(x)}{W}$$

$$A' = \frac{-e^{2x}\cos(e^{-x})}{e^{3x}} \qquad B' = \frac{e^x \cos(e^{-x})}{e^{3x}}$$

$$A' = -e^{-x}\cos(e^{-x}) \qquad B' = e^{-2x}\cos(e^{-x})$$

$$\Rightarrow \quad A = \int -e^{-x}\cos(e^{-x})\,dx + k_1 \qquad B = \int e^{-2x}\cos(e^{-x})\,dx + k_2$$

Put $\quad e^{-x} = t \quad \therefore \quad -e^{-x}\,dx = dt$

$$A = \int \cos t\,dt + k_1 \qquad B = \int -t\cos t\,dt + k_2$$

$$A = \sin t + k_1 \qquad B = -t\sin t - \int \sin t\,(-1)\,dt + k_2.$$

$$B = -t\sin t - \cos t + k_2$$

ie., $\quad A = \sin(e^{-x}) + k_1 \qquad B = -e^{-x}\sin(e^{-x}) - \cos(e^{-x}) + k_2$

Substituting these in $y = A e^{x} + B e^{2x}$ we have,

$$y = \left\{ \sin(e^{-x}) + k_1 \right\} e^{x} + \left\{ -e^{-x}\sin(e^{-x}) - \cos(e^{-x}) + k_2 \right\} e^{2x}$$

Thus $\quad \mathbf{y = k_1 e^{x} + k_2 e^{2x} - e^{2x}\cos(e^{-x})}$

---

16. *Solve* $(D^2 + 3D + 2)\,y = e^{e^x}$ *by the method of variation of parameters.*

>> A.E is $m^2 + 3m + 2 = 0$ or $(m+1)(m+2) = 0 \Rightarrow m = -1, -2$

$\therefore \quad y_c = c_1 e^{-x} + c_2 e^{-2x}$

$y = A e^{-x} + B e^{-2x}$ be the complete solution of the given equation where A and B are functions of x to be found.

We have $\quad y_1 = e^{-x} \quad ; \quad y_2 = e^{-2x}$

$y_1' = -e^{-x} \quad ; \quad y_2' = -2e^{-2x}$

$W = y_1 y_2' - y_2 y_1' = -e^{-3x}$. Also $\phi(x) = e^{e^x}$

$$A' = \frac{-y_2\,\phi(x)}{W} \qquad B' = \frac{y_1\,\phi(x)}{W}$$

$$A' = \frac{-e^{-2x} e^{e^x}}{-e^{-3x}} \qquad B' = \frac{e^{-x} e^{e^x}}{-e^{-3x}}$$

$$A' = e^{x} e^{e^x} \qquad B' = -e^{2x} e^{e^x}$$

$\Rightarrow \quad A = \int e^{x} e^{e^x}\,dx + k_1 \qquad B = -\int e^{2x} e^{e^x}\,dx + k_2$

Put $\quad e^{x} = t \quad \therefore \; e^{x}\,dx = dt$

$$A = \int e^{t}\,dt + k_1 \qquad B = -\int t\,e^{t}\,dt + k_2$$

$$A = e^{t} + k_1 \qquad B = -(t e^{t} - e^{t}) + k_2$$

$$A = e^{e^x} + k_1 \qquad B = e^{e^x}(1 - e^{x}) + k_2$$

Substituting these in $y = A e^{-x} + B e^{-2x}$ we have,

$$y = \left\{ e^{e^x} + k_1 \right\} e^{-x} + \left\{ e^{e^x}(1 - e^{x}) + k_2 \right\} e^{-2x}$$

Thus $\quad \mathbf{y = k_1 e^{-x} + k_2 e^{-2x} + e^{-2x} e^{e^x}}$

An Illustrate example on the application of the method for a third order equation :

$$(D^3 + 6D^2 + 11D + 6)\,y = 24\,e^x$$

>> We have $y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$ *(Refer problem-2 in Unit-II)*

$$y = A\,e^{-x} + B\,e^{-2x} + C\,e^{-3x} \qquad \ldots (1)$$

be the complete solution of the given equation where A, B, C are functions of x to be found.

$$y' = (-A\,e^{-x} - 2B\,e^{-2x} - 3C\,e^{-3x}) + (A'\,e^{-x} + B'\,e^{-2x} + C'\,e^{-3x})$$

We shall find A, B, C such that

$$A'\,e^{-x} + B'\,e^{-2x} + C'\,e^{-3x} = 0 \qquad \ldots (2)$$

Hence $y' = -(A\,e^{-x} + 2B\,e^{-2x} + 3C\,e^{-3x})$ $\qquad \ldots$ (a)

$$y'' = (A\,e^{-x} + 4B\,e^{-2x} + 9C\,e^{-3x}) - (A'\,e^{-x} + 2B'\,e^{-2x} + 3C'\,e^{-3x})$$

Let us find A, B, C such that

$$A'\,e^{-x} + 2B'\,e^{-2x} + 3C'\,e^{-3x} = 0 \qquad \ldots (3)$$

$$\therefore \quad y'' = (A\,e^{-x} + 4B\,e^{-2x} + 9C\,e^{-3x}) \qquad \ldots (b)$$

Hence $y''' = -(A\,e^{-x} + 8B\,e^{-2x} + 27\,C\,e^{-3x}) + (A'\,e^{-x} + 4B'\,e^{-2x} + 9C'\,e^{-3x}) \ldots$ (4)

Now consider the given equation

$$y''' + 6y'' + 11y' + 6y = 24\,e^x$$

This equation as a consequence of (1), (a), (b) and (4) becomes,

$$-(A\,e^{-x} + 8B\,e^{-2x} + 27C\,e^{-3x}) + (A'\,e^{-x} + 4B'\,e^{-2x} + 9C'\,e^{-3x})$$
$$+\ (6A\,e^{-x} + 24\,B\,e^{-2x} + 54C\,e^{-3x}) - 11\,(A\,e^{-x} + 2B\,e^{-2x} + 3C\,e^{-3x})$$
$$+\ 6\,(A\,e^{-x} + B\,e^{-2x} + C\,e^{-3x}) = 24\,e^x$$

ie., $\quad A'\,e^{-x} + 4B'\,e^{-2x} + 9C'\,e^{-3x} = 24\,e^x$ $\qquad \ldots$ (5)

Now let us solve (2), (3), (5) to find A', B', C'

$$A'\,e^{-x} + B'\,e^{-2x} + C'\,e^{-3x} = 0 \qquad \ldots (2)$$

$$A'\,e^{-x} + 2B'\,e^{-2x} + 3C'\,e^{-3x} = 0 \qquad \ldots (3)$$

$$A'\,e^{-x} + 4B'\,e^{-2x} + 9C'\,e^{-3x} = 24\,e^x \qquad \ldots (5)$$

(3)–(2) : $B'\,e^{-2x} + 2C'\,e^{-3x} = 0$ $\qquad \ldots$ (6)

(5)–(3) : $2B'e^{-2x} + 6C'e^{-3x} = 24e^{x}$

or $\quad B'e^{-2x} + 3C'e^{-3x} = 12e^{x}$ ... (7)

Now (7)–(6) : $C'e^{-3x} = 12e^{x}$ or $C' = 12e^{4x} \Rightarrow C = 3e^{4x} + k_1$

From (6) $B' = -24e^{3x} \Rightarrow B = -8e^{3x} + k_2$

From (2) : $A' = 12e^{2x} \Rightarrow A = 6e^{2x} + k_3$

Substituting $A(x)$, $B(x)$, $C(x)$ in (1) we get,

$$y = (6e^{2x} + k_3)e^{-x} + (-8e^{3x} + k_2)e^{-2x} + (3e^{4x} + k_1)e^{-3x}$$

Thus $\mathbf{y = k_1 e^{-3x} + k_2 e^{-2x} + k_3 e^{-x} + e^{x}}$

EXERCISES

Solve the following equations by the method of variation of parameters.

1. $y'' + y = \sec^3 x$
2. $(D^2 + 1)y = \operatorname{cosec} x$
3. $(D^2 + 9)y = 9\sec 3x \tan 3x$
4. $(D^2 + 3D + 2)y = e^{-x}$
5. $y'' - 2y' + y = e^{x}/x^5$
6. $y'' + 2y' + y = \log x/e^{x}$
7. $(D^2 - 2D + 2)y = e^{x}\tan x$
8. $\dfrac{d^2y}{dx^2} - y = \dfrac{1}{(1 + e^{-x})^2}$
9. $y'' - y = xe^{2x}$
10. $x'''(t) - 6x''(t) + 11x'(t) - 6x(t) = e^{2t}$

ANSWERS

1. $y = k_1 \cos x + k_2 \sin x + \sec x/2$
2. $y = k_1 \cos x + k_2 \sin x - x\cos x + \sin x \log(\sin x)$
3. $y = k_1 \cos 3x + k_2 \sin 3x + 3x\cos 3x + \sin 3x \log(\sec 3x)$
4. $y = k_1 e^{-x} + k_2 e^{-2x} + xe^{-x}$
5. $y = (k_1 + k_2 x)e^{x} + e^{x}/12x^3$
6. $y = (k_1 + k_2 x)e^{-x} + x^2 e^{-x}(2\log x - 3)/4$

7. $y = e^x(k_1 \cos x + k_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$

8. $y = k_1 e^x + k_2 e^{-x} + e^{-x}\log(1+e^x) - 1$

9. $k_1 e^x + k_2 e^{-x} + e^{2x}(3x-4)/9$

10. $x(t) = k_1 e^t + k_2 e^{2t} + k_3 e^{3t} - te^{2t}$

3.3 Differential Equation with variable coefficients reducible to equation with constant coefficients

In this article we consider differential equation in some specific forms involving variable coefficients. These can be solved by reducing into an equation with constant coefficients by a specific substitution. The theory is illustrated by considering a second order equation and we can conveniently extend for higher order equations also. We first discuss Legendre's linear equation and subsequently discuss Cauchy's linear equation as a particular case.

3.31 Legendre's linear equation

An equation of the form,

$$a_0(ax+b)^2 y'' + a_1(ax+b)y' + a_2 y = \phi(x) \qquad \ldots (1)$$

is called *Legendre's linear equation* of second order. We use a substitution to reduce the same into a D.E with constant coefficients.

Put $t = \log(ax+b)$ or $e^t = (ax+b)$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{a}{ax+b} \quad \text{or} \quad (ax+b)\frac{dy}{dx} = a\frac{dy}{dt}$$

Let $D = \dfrac{d}{dt}$ so that we have

$$(ax+b)y' = a\,Dy \qquad \ldots (2)$$

Differentiating (2) w.r.t. x again we get,

$$(ax+b)y'' + a y' = a\cdot\frac{d}{dx}\left(\frac{dy}{dt}\right) = a\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

ie., $$(ax+b)y'' + a y' = a\frac{d^2y}{dt^2}\cdot\frac{a}{ax+b}$$

ie., $$(ax+b)^2 y'' + a(ax+b)y' = a^2\frac{d^2y}{dt^2}$$

ie., $$(ax+b)^2 y'' + a\cdot a\,Dy = a^2 D^2 y$$

or $\quad (ax+b)^2 y'' = a^2 (D^2 - D) y$

Hence we have $\quad (ax+b)^2 y'' = a^2 D(D-1) y \quad \ldots (3)$

Using (2) and (3) in (1) we have,

$$[a_0 \cdot a^2 D(D-1) + a_1 \cdot a D + a_2] y = F(t) \quad \ldots (4)$$

where the R.H.S of (1) being $\phi(x) = \phi\left(\dfrac{e^t - b}{a}\right) = F(t)$

Clearly (4) is a linear differential equation with constant coefficients of the form

$$f(D) y = F(t) \text{ where } D = \frac{d}{dt}$$

We can solve this equation as already discussed to obtain the solution y in terms of t which can be converted back to x.

Note : We can further show that,

$$(ax+b)^3 y''' = a^3 D(D-1)(D-2) y$$

$$(ax+b)^4 \ y'''' = a^4 D(D-1)(D-2)(D-3) y \text{ etc.}$$

3.32 Cauchy's linear equation

This is a particular case of Legendre's linear equation when $a = 1$ and $b = 0$.

That is,

$$a_0 x^2 y'' + a_1 x y' + a_2 y = \phi(x) \quad \ldots (1)$$

This is called *Cauchy's homogeneous linear equation* of second order.
As discussed earlier when we substitute $t = \log x$ or $x = e^t$ we get $xy' = Dy$, $x^2 y'' = D(D-1) y$ so that (1) reduces to the form

$$[a_0 \ D(D-1) + a_1 D + a_2] y = \phi(e^t) = F(t) \ (say)$$

ie., $\quad f(D) y = F(t) \quad \ldots (2)$

(2) is a linear d.e with constant coefficients where $D = \dfrac{d}{dt}$

Note : *Obviously we have* $x^3 y''' = D(D-1)(D-2) y$ *and so on.*

Working procedure for problems

- We ensure that the equation is in the standard form of Legendre's or Cauchy's linear equation. Sometimes we have to employ simple techniques to bring it into the standard form.
- (a) We take the substitution $t = \log(ax+b)$ or $e^t = (ax+b)$ in the case of Legendre's equation.

We assume the results :

$(ax+b)y' = aDy,\ (ax+b)^2 y'' = a^2 D(D-1)y$ etc. where $D = \dfrac{d}{dt}$ to obtain a linear differential equation with constant coefficients.

(b) We take the substitution $t = \log x$ or $x = e^t$ in the case of Cauchy's equation.
We assume the results :

$xy' = Dy,\ x^2 y'' = D(D-1)y,\ x^3 y''' = D(D-1)(D-2)y$ etc.

where $D = \dfrac{d}{dt}$ to obtain a linear equation with constant coefficients.

- We solve the linear differential equation with constant coefficients to obtain $y = y_c + y_p$ in terms of t
- We substitute for t and present the solution y in terms of x

WORKED PROBLEMS

Problems on Cauchy's linear equation

17. Solve $x^2 \dfrac{d^2 y}{dx^2} - 3x \dfrac{dy}{dx} + 4y = (1+x)^2$

\>> We have $x^2 y'' - 3xy' + 4y = (1+x)^2$ ...(1)

Put $t = \log x$ or $e^t = x$.

Then we have $xy' = Dy,\ x^2 y'' = D(D-1)y$ where $D = \dfrac{d}{dt}$

Hence (1) becomes, $[(D(D-1) - 3D + 4]y = (1+e^t)^2$

ie., $(D^2 - 4D + 4)y = 1 + 2e^t + e^{2t}$

A.E is $m^2 - 4m + 4 = 0$ or $(m-2)^2 = 0 \Rightarrow m = 2, 2$

$\therefore \quad y_c = (c_1 + c_2 t)e^{2t}$

$$y_p = \frac{1}{D^2-4D+4} + \frac{2e^t}{D^2-4D+4} + \frac{e^{2t}}{D^2-4D+4} = p_1 + p_2 + p_3 \ (say)$$

$$p_1 = \frac{e^{0t}}{D^2-4D+4} = \frac{e^{0t}}{0-0+4} = \frac{1}{4}$$

$$p_2 = \frac{2e^t}{D^2-4D+4} = \frac{2e^t}{1-4+4} = 2e^t$$

$$p_3 = \frac{e^{2t}}{D^2-4D+4} = \frac{e^{2t}}{4-8+4} \ (Dr. = 0)$$

$$p_3 = t \cdot \frac{e^{2t}}{2D-4} = t \cdot \frac{e^{2t}}{4-4} \quad (Dr. = 0)$$

$$\therefore \ p_3 = t^2 \frac{e^{2t}}{2}$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2 + p_3$

Thus $\mathbf{y = (c_1 + c_2 \log x)\, x^2 + \frac{1}{4} + 2x + \frac{x^2 (\log x)^2}{2}}$

---

18. *Solve :* $x^2 y'' + x y' + 9y = 3x^2 + \sin(3 \log x)$

>> Put $t = \log x$ or $e^t = x$

Then we have $x y' = Dy$, $x^2 y'' = D(D-1)\, y$ where $D = \frac{d}{dt}$

The given equation becomes,

$$[D(D-1) + D + 9]\, y = 3e^{2t} + \sin 3t$$

ie., $(D^2 + 9)\, y = 3e^{2t} + \sin 3t$

A.E is $m^2 + 9 = 0 \Rightarrow m = \pm 3i$

$\therefore$ $y_c = c_1 \cos 3t + c_2 \sin 3t$

$$y_p = \frac{3e^{2t}}{D^2+9} + \frac{\sin 3t}{D^2+9} = p_1 + p_2 \ (say)$$

$$p_1 = \frac{3e^{2t}}{D^2+9} = \frac{3e^{2t}}{2^2+9} = \frac{3e^{2t}}{13}$$

$$p_2 = \frac{\sin 3t}{D^2+9} = \frac{\sin 3t}{-3^2+9} \; (\, Dr. = 0\,)$$

$$= t \cdot \frac{\sin 3t}{2D} = \frac{t}{2} \int \sin 3t \, dt = \frac{-t \cos 3t}{6}$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2$

Thus $\quad y = c_1 \cos(3 \log x) + c_2 \sin(3 \log x) + \frac{3x^2}{13} - \frac{\log x \cdot \cos(3 \log x)}{6}$

---

19. *Solve :* $x \frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} = \frac{1}{x}$

\>> We have $x y''' + y'' = \frac{1}{x}$ and multiplying by x^2 we get,

$$x^3 y''' + x^2 y'' = x \qquad \ldots (1)$$

Put $t = \log x$ or $e^t = x$. Then we have

$$x y' = Dy, \; x^2 y'' = D(D-1)y, \; x^3 y''' = D(D-1)(D-2)y \quad \text{where } D = \frac{d}{dt}$$

Hence (1) becomes $[\,D(D-1)(D-2) + D(D-1)\,]\, y = e^t$

A.E is $\quad m(m-1)(m-2) + m(m-1) = 0$

ie., $\quad m(m-1)[\,(m-2)+1\,] = 0$

or $\quad m(m-1)^2 = 0 \quad \Rightarrow \quad m = 0, 1, 1$

$\therefore \quad y_c = c_1 + (c_2 + c_3 t)\, e^t$

$$y_p = \frac{e^t}{D^3 - 2D^2 + D} = \frac{e^t}{1-2+1} \qquad (\, Dr. = 0\,)$$

$$= t \cdot \frac{e^t}{3D^2 - 4D + 1} = t \cdot \frac{e^t}{3-4+1} \quad (\, Dr. = 0\,)$$

$$= t^2 \frac{e^t}{6D-4} = \frac{t^2 e^t}{2}$$

Complete solution : $y = y_c + y_p$

Thus $\quad y = c_1 + (c_2 + c_3 \log x)\, x + \frac{x(\log x)^2}{2}$

---

20. *Solve :* $x\dfrac{d^2y}{dx^2} - \dfrac{2y}{x} = x + \dfrac{1}{x^2}$

>> We have $x\,y'' - \dfrac{2y}{x} = x + \dfrac{1}{x^2}$ and multiplying by x we get,

$$x^2 y'' - 2y = x^2 + \frac{1}{x} \qquad \ldots (1)$$

Put $t = \log x$ or $e^t = x$

Then we have, $x\,y' = Dy,\ x^2 y'' = D(D-1)y$ where $D = \dfrac{d}{dt}$

Hence (1) becomes, $[\,D(D-1)-2\,]\,y = e^{2t} + e^{-t}$

ie., $(D^2 - D - 2)\,y = e^{2t} + e^{-t}$

A.E is $m^2 - m - 2 = 0$ or $(m-2)(m+1) = 0 \Rightarrow m = 2, -1$

$\therefore \quad y_c = c_1 e^{2t} + c_2 e^{-t}$

$$y_p = \frac{e^{2t}}{D^2 - D - 2} + \frac{e^{-t}}{D^2 - D - 2} = p_1 + p_2\ (say)$$

$$p_1 = \frac{e^{2t}}{D^2 - D - 2} = \frac{e^{2t}}{4-2-2}\ (Dr. = 0)$$

$$= t\cdot\frac{e^{2t}}{2D-1} = t\cdot\frac{e^{2t}}{3} = \frac{t\,e^{2t}}{3}$$

$$p_2 = \frac{e^{-t}}{D^2 - D - 2} = \frac{e^{-t}}{1+1-2}\ (Dr. = 0)$$

$$= t\cdot\frac{e^{-t}}{2D-1} = t\cdot\frac{e^{-t}}{-3} = -\frac{t\,e^{-t}}{3}$$

Complete solution : $y = y_c + y_p$, where $y_p = p_1 + p_2$

Thus $\quad \mathbf{y = c_1 x^2 + \dfrac{c_2}{x} + \dfrac{\log x}{3}\left(x^2 - \dfrac{1}{x}\right)}$

---

21. *Solve :* $x^3\dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 8y = 65\cos(\log x)$

>> Put $t = \log x$ or $x = e^t$ Then we have,

$x\,y' = D\,y,\ x^2 y'' = D(D-1)\,y,\ x^3 y''' = D(D-1)(D-2)\,y$ where $D = \dfrac{d}{dt}$

Hence the given d.e becomes,

$$[D(D-1)(D-2)+3D(D-1)+D+8]\,y = 65\cos t$$

ie., $[D^3-3D^2+2D+3D^2-3D+D+8]\,y = 65\cos t$

or $(D^3+8)\,y = 65\cos t$

A.E is given by $m^3+8 = 0$

or $(m+2)(m^2-2m+4) = 0$

or $m = -2$ and $m^2-2m+4 = 0$

By solving $m^2-2m+4 = 0$ we have,

$$m = \frac{2\pm\sqrt{4-16}}{2} = \frac{2\pm 2i\sqrt{3}}{2} = 1\pm i\sqrt{3}$$

$\therefore$ $y_c = c_1 e^{-2t}+e^t\left\{c_2\cos(\sqrt{3}\,t)+c_3\sin(\sqrt{3}\,t)\right\}$

Also $y_p = \dfrac{65\cos t}{D^3+8}$. We replace D^2 by $-1^2 = -1$

$$y_p = \frac{65\cos t}{-D+8} = \frac{65(8+D)\cos t}{64-D^2}$$

$$y_p = \frac{65(8\cos t-\sin t)}{65} = 8\cos t-\sin t$$

Complete solution : $y = y_c+y_p$

Thus, $y = \dfrac{c_1}{x^2}+x\left\{c_2\cos(\sqrt{3}\log x)+c_3\sin(\sqrt{3}\log x)\right\} + 8\cos(\log x)-\sin(\log x)$

---

22. *Solve :* $x^2\dfrac{d^3y}{dx^3} + 3x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2\log x$

>> Multiplying the given equation by x we have,

$$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = x^3\log x \qquad \ldots(1)$$

Put $t = \log x$ or $x = e^t$. Then we have,

$xy' = Dy$, $x^2y'' = D(D-1)y$, $x^3y''' = D(D-1)(D-2)y$ where $D = \dfrac{d}{dt}$

Hence (1) becomes,

$$[\,D\,(\,D-1\,)\,(\,D-2\,)+3D\,(\,D-1\,)+D\,] = e^{3t}\,t$$

i.e., $D^3\,y = 0$

A.E is $m^3 = 0$ and hence $m = 0, 0, 0$

$$y_c = (\,c_1 + c_2\,t + c_3\,t^2\,)\,e^{0t} \quad \text{or} \quad y_c = c_1 + c_2\,t + c_3\,t^2$$

$$y_p = \frac{e^{3t}\,t}{D^3} = e^{3t}\cdot\frac{t}{(\,D+3\,)^3} = e^{3t}\cdot\frac{t}{D^3+9D^2+27D+27}$$

P.I is found by division.

	$t/27 - 1/27$
$27+27D+9D^2+D^3$	t
	$t+1$
	-1
	-1
	0

$$y_p = e^{3t}\cdot\frac{(\,t-1\,)}{27}$$

Complete solution : $y = y_c + y_p$

Thus $$y = \left\{\,c_1 + c_2\log x + c_3\,(\,\log x\,)^2\right\} + \frac{x^3\,(\,\log x - 1\,)}{27}$$

---

23. *Solve :* $x^2\,y'' - x\,y' + 2y = x\sin(\,\log x\,)$

>> Put $t = \log x$ or $e^t = x$. Then we have

$$x\,y' = Dy,\; x^2\,y'' = D\,(\,D-1\,)\,y \text{ where } D = \frac{d}{dt}$$

Hence the given equation becomes

$$[\,D\,(\,D-1\,)-D+2\,]\,y = e^t\sin t$$

ie., $(\,D^2-2D+2\,)\,y = e^t\sin t$

A.E is $m^2-2m+2 = 0$ and by solving,

$$m = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm 2i}{2} = 1\pm i$$

$\therefore \quad y_c = e^t (c_1 \cos t + c_2 \sin t)$

$$y_p = \frac{e^t \sin t}{D^2 - 2D + 2} \quad \text{Now } D \to D+1$$

$$y_p = e^t \cdot \frac{\sin t}{(D+1)^2 - 2(D+1) + 2} = e^t \cdot \frac{\sin t}{D^2+1}$$

$$= e^t \cdot \frac{\sin t}{-1^2+1} \quad (Dr. = 0)$$

$$= e^t \cdot t \frac{\sin t}{2D} = e^t \cdot t \int \frac{\sin t}{2} dt = \frac{-e^t \, t \cos t}{2}$$

Complete solution : $y = y_c + y_p$

Thus $\quad \boldsymbol{y = x \left\{ c_1 \cos(\log x) + c_2 \sin(\log x) \right\} - \frac{x \log x \cos(\log x)}{2}}$

---

24. *Solve :* $x^4 \frac{d^3 y}{dx^3} + 2x^3 \frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + x y = \sin(\log x)$

>> Dividing the given equation throughout by x we get,

$$x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \frac{1}{x} \sin(\log x) \quad \ldots (1)$$

Put $t = \log x$ or $e^t = x$. Then we have,

$$x y' = Dy, \; x^2 y'' = D(D-1) y, \; x^3 y''' = D(D-1)(D-2) y \quad \text{where } D = \frac{d}{dt}$$

Hence (1) becomes

$$[D(D-1)(D-2) + 2D(D-1) - D + 1] y = e^{-t} \sin t$$

ie., $\quad [D^3 - 3D^2 + 2D + 2D^2 - 2D - D + 1] y = e^{-t} \sin t$

ie., $\quad [D^3 - D^2 - D + 1] y = e^{-t} \sin t$

A.E is $\quad m^3 - m^2 - m + 1 = 0$

or $\quad m^2 (m-1) - 1(m-1) = 0$ or $(m-1)(m^2-1) = 0$

ie., $\quad (m-1)^2 (m+1) = 0 \Rightarrow m = 1, 1, -1$

$\therefore \quad y_c = (c_1 + c_2 t) e^t + c_3 e^{-t}$

$$y_p = \frac{e^{-t}\sin t}{(D-1)^2(D+1)} \quad \text{Now } D \to D-1$$

$$y_p = e^{-t}\cdot\frac{\sin t}{(D-2)^2 D} = e^{-t}\cdot\frac{\sin t}{D^2\cdot D - 4D^2 + 4D} \quad \text{Now } D^2 \to -1$$

$$y_p = e^{-t}\cdot\frac{\sin t}{3D+4} = e^{-t}\cdot\frac{(3D-4)\sin t}{(3D-4)(3D+4)}$$

$$= e^{-t}\cdot\frac{3\cos t - 4\sin t}{9D^2-16} = e^{-t}\cdot\frac{3\cos t - 4\sin t}{-25}$$

$$y_p = \frac{e^{-t}}{25}(4\sin t - 3\cos t)$$

Complete solution : $y = y_c + y_p$

Thus $\quad y = (c_1 + c_2\log x)\,x + \dfrac{c_3}{x} + \dfrac{1}{25x}[4\sin(\log x) - 3\cos(\log x)]$

---

25. *Solve :* $x^2 y'' - x y' + y = x^2 \log x$

>> Put $t = \log x$ or $x = e^t$. Then we have,

$$x y' = Dy,\ x^2 y'' = D(D-1)y, \text{ where } D = \frac{d}{dt}$$

Hence the given equation becomes,

$$[D(D-1) - D + 1]\,y = e^{2t}\,t$$

ie., $\quad (D^2 - 2D + 1)\,y = e^{2t}\cdot t$

A.E is $m^2 - 2m + 1 = 0$ or $(m-1)^2 = 0 \Rightarrow m = 1, 1$

$\therefore \quad y_c = (c_1 + c_2 t)\,e^t$

$$y_p = \frac{e^{2t}\,t}{(D-1)^2} \quad \text{Now } D \to D+2$$

$$= e^{2t}\cdot\frac{t}{(D+1)^2} = e^{2t}\cdot\frac{t}{D^2+2D+1} = e^{2t}\cdot\frac{t}{1+2D+D^2}$$

We have to employ division.

$$\begin{array}{r|l} & \quad t-2 \\ \hline 1+2D+D^2 & \quad t \\ & \quad t+2 \\ \hline & \quad -2 \\ & \quad -2 \\ \hline & \quad 0 \end{array}$$

Quotient $= (t-2)$
and $y_p = e^{2t}(t-2)$

Complete solution : $y = y_c + y_p$

Thus $y = (c_1 + c_2 \log x)\, x + x^2 (\log x - 2)$

---

26. *Solve :* $x^2 \dfrac{d^2 y}{dx^2} - (2m-1)\, x \dfrac{dy}{dx} + (m^2 + n^2)\, y = n^2 x^m \log x$

>> Put $t = \log x$ or $e^t = x$. Then we have

$$x y' = Dy,\ x^2 y'' = D(D-1)\, y \text{ where } D = \frac{d}{dt}$$

Hence the given equation becomes

$$[D(D-1) - (2m-1)D + (m^2+n^2)]\, y = n^2 e^{mt}\, t$$

ie., $$[D^2 - 2mD + (m^2+n^2)]\, y = n^2 e^{mt}\, t$$

A.E is $p^2 - 2mp + (m^2+n^2) = 0$ and we solve for p.

(We have replaced D by p to write the A.E since m is involved in the example.)

$$p = \frac{2m \pm \sqrt{4m^2 - 4(m^2+n^2)}}{2} = \frac{2m \pm 2in}{2} = m \pm in$$

$\therefore$ $$y_c = e^{mt}(c_1 \cos nt + c_2 \sin nt)$$

$$y_p = \frac{n^2 e^{mt}\, t}{D^2 - 2mD + (m^2+n^2)} \quad \text{Now } D \to D + m$$

$$= e^{mt} \cdot \frac{n^2 t}{(D+m)^2 - 2m(D+m) + (m^2+n^2)}$$

$$= e^{mt} \cdot \frac{n^2 t}{D^2 + n^2}$$

We employ division now.

$$n^2 + D^2 \,\big)\, n^2 t \,\big(\, t$$
$$\underline{n^2 t + 0}$$
$$0$$

Quotient $= t$ and $y_p = e^{mt} \cdot t$

Complete solution : $y = y_c + y_p$.

Thus $y = x^m [c_1 \cos(n \log x) + c_2 \sin(n \log x)] + x^m \log x$

---

27. *Solve the Cauchy's homogeneous linear equation*

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = \log x \sin(\log x)$$

>> Put $t = \log x$ or $e^t = x$. Then we have,

$$x y' = Dy, \; x^2 y'' = D(D-1)y \text{ where } D = \frac{d}{dt}$$

Hence the given equation becomes

$$[D(D-1)+D+1]y = t\sin t$$

ie., $(D^2+1)y = t\sin t$

A.E is $m^2+1 = 0 \Rightarrow m = \pm i$

$\therefore \quad y_c = c_1 \cos t + c_2 \sin t$

$$y_p = \frac{t\sin t}{D^2+1} \text{ and we use } \frac{tV}{f(D)} = \left[t - \frac{f'(D)}{f(D)}\right]\frac{V}{f(D)}$$

$$y_p = \left[t - \frac{2D}{D^2+1}\right]\frac{\sin t}{D^2+1} \qquad D^2 \to -1 \;(Dr. = 0)$$

$$y_p = \left[t - \frac{2D}{D^2+1}\right] t \cdot \frac{\sin t}{2D}$$

$$y_p = \left[t - \frac{2D}{D^2+1}\right]\left(\frac{-t\cos t}{2}\right)$$

$$y_p = \frac{-t^2\cos t}{2} + \frac{D(t\cos t)}{D^2+1}$$

$$y_p = \frac{-t^2\cos t}{2} - \frac{t\sin t}{D^2+1} + \frac{\cos t}{D^2+1} \quad \text{But } \frac{t\sin t}{D^2+1} = y_p$$

$$\therefore \quad y_p = \frac{-t^2\cos t}{2} - y_p + t \cdot \frac{\cos t}{2D} \quad \text{or } 2y_p = \frac{-t^2\cos t}{2} + \frac{t\sin t}{2}$$

ie., $2y_p = \frac{t}{2}(\sin t - t\cos t)$ or $y_p = \frac{t}{4}(\sin t - t\cos t)$

Complete solution : $y = y_c + y_p$

$$\mathbf{y = c_1 \cos(\log x) + c_2 \sin(\log x) + \frac{\log x}{4}[\sin(\log x) - \log x \cos(\log x)]}$$

- -

28. *Solve :* $x^2 D^2 y - 3x\, Dy + 5y = x^2 \sin(\log x)$

>> We have $x^2 y'' - 3x y' + 5y = x^2 \sin(\log x)$ . . . (1)

Put $t = \log x$ or $e^t = x$. Then we have

$$x y' = Dy,\ x^2 y'' = D(D-1)y \text{ where } D = \frac{d}{dt}$$

Hence (1) becomes, $[D(D-1) - 3D + 5]y = e^{2t} \sin t$

ie., $(D^2 - 4D + 5)y = e^{2t} \sin t$

A.E is $m^2 - 4m + 5 = 0$ and by solving,

$$m = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$\therefore$ $y_c = e^{2t}(c_1 \cos t + c_2 \sin t)$

$$y_p = \frac{e^{2t} \sin t}{D^2 - 4D + 5} \quad \text{Now } D \to D+2$$

$$y_p = e^{2t} \frac{\sin t}{(D+2)^2 - 4(D+2) + 5} = e^{2t} \cdot \frac{\sin t}{D^2 + 1}$$

$$y_p = e^{2t} \cdot t \cdot \frac{\sin t}{2D} = \frac{-e^{2t}\, t \cos t}{2}$$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = x^2 [c_1 \cos(\log x) + c_2 \sin(\log x)] - \frac{x^2 \log x \cos(\log x)}{2}}$

Problems on Legendre's linear equation

29. *Solve the Legendre's form of linear equation*

$$(1+x)^2 \frac{d^2 y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin 2[\log(1+x)]$$

>> Put $t = \log(1+x)$ or $e^t = 1+x$ Then we have

$$(1+x)\frac{dy}{dx} = 1 \cdot Dy,\ (1+x)^2 \frac{d^2 y}{dx^2} = 1^2 D(D-1)y \ ; \text{ where } D = \frac{d}{dt}$$

Hence the given D.E becomes,

$$[D(D-1) + D + 1]y = \sin 2t$$

ie., $(D^2 + 1)y = \sin 2t$

A.E is $m^2+1 = 0 \Rightarrow m = \pm i$

$\therefore \quad y_c = c_1 \cos t + c_2 \sin t$

$$y_p = \frac{\sin 2t}{D^2+1} = \frac{\sin 2t}{-2^2+1} = \frac{-\sin 2t}{3}$$

Complete solution : $y = y_c + y_p$

Thus $\quad \boldsymbol{y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \frac{\sin 2[\log(1+x)]}{3}}$

---

30. *Solve* : $(2x+1)^2 y'' - 6(2x+1)y' + 16y = 8(2x+1)^2$

>> Put $\quad t = \log(2x+1)$ or $e^t = 2x+1$ Then we have,

$$(2x+1)y' = 2 \cdot Dy, \; (2x+1)^2 y'' = 2^2 D(D-1)y, \text{ where } D = \frac{d}{dt}$$

Hence the given equation becomes

$$[4D(D-1) - 6 \cdot 2D + 16]y = 8e^{2t}$$

ie., $\quad (D^2 - D - 3D + 4)y = 2e^{2t}$, on dividing by 4.

ie., $\quad (D^2 - 4D + 4)y = 2e^{2t}$

A.E is $m^2 - 4m + 4 = 0$ or $(m-2)^2 = 0 \Rightarrow m = 2, 2$

$\therefore \quad y_c = (c_1 + c_2 t)e^{2t}$

$$y_p = \frac{2e^{2t}}{(D-2)^2} = \frac{2e^{2t}}{(2-2)^2} \quad (Dr. = 0)$$

$$= t\frac{2e^{2t}}{2(D-2)} = t\frac{e^{2t}}{(2-2)} \quad (Dr. = 0)$$

$$y_p = t^2\frac{e^{2t}}{1} = t^2 e^{2t}$$

Complete solution : $y = y_c + y_p$

Thus $\quad \boldsymbol{y = [c_1 + c_2 \log(2x+1)](2x+1)^2 + \{\log(2x+1)\}^2 (2x+1)^2}$

---

31. *Solve :* $(2x+1)^2 y'' - 2(2x+1)y' - 12y = 6x+5$

>> Put $t = \log(2x+1)$ or $e^t = 2x+1$ Then we have

$$(2x+1)y' = 2\cdot Dy,\ (2x+1)^2 y'' = 2^2 D(D-1)y \text{ where } D = \frac{d}{dt}$$

Hence the given equation becomes,

$$[4D(D-1) - 4D - 12]y = 6\left(\frac{e^t-1}{2}\right)+5$$

ie., $4(D^2-2D-3)y = 3e^t+2$

or $(D^2-2D-3)y = \frac{3}{4}e^t + \frac{1}{2}$

A.E is $m^2-2m-3=0$ or $(m-3)(m+1)=0 \Rightarrow m = 3, -1$

$\therefore$ $y_c = c_1 e^{3t} + c_2 e^{-t}$

$$y_p = \frac{3}{4}\frac{e^t}{D^2-2D-3} + \frac{1}{2}\cdot\frac{e^{0t}}{D^2-2D-3}$$

$$y_p = \frac{3}{4}\frac{e^t}{1-2-3} + \frac{1}{2}\frac{e^{0t}}{0-0-3}$$

$$y_p = \frac{-3e^t}{16} - \frac{1}{6}$$

Complete solution : $y = y_c + y_p$

Thus $\quad y = c_1(2x+1)^3 + \frac{c_2}{(2x+1)} - \frac{3(2x+1)}{16} - \frac{1}{6}$

---

32. *Solve :* $(3x+2)^2 y'' + 3(3x+2)y' - 36y = 8x^2+4x+1$

>> Put $t = \log(3x+2)$ or $e^t = 3x+2$ Then we have,

$$(3x+2)y' = 3\cdot Dy,\ (3x+2)^2 y'' = 3^2\cdot D(D-1)y \quad \text{where } D = \frac{d}{dt}$$

Further $x = \frac{1}{3}(e^t-2)$

Hence the given equation becomes

$$[9D(D-1) + 9D - 36]y = 8\cdot\frac{1}{9}(e^t-2)^2 + 4\cdot\frac{1}{3}(e^t-2)+1$$

ie., $9(D^2-4)y = \frac{8}{9}(e^{2t}-4e^t+4)+\frac{4}{3}(e^t-2)+1$

ie., $9(D^2-4)y = \frac{8}{9}e^{2t}-\frac{20}{9}e^t+\frac{17}{9}$

or $(D^2-4)y = \frac{1}{81}(8e^{2t}-20e^t+17)$

A.E is $m^2-4 = 0 \Rightarrow m = \pm 2$

$\therefore \quad y_c = c_1 e^{2t}+c_2 e^{-2t}$

$$y_p = \frac{1}{81}\left[\frac{8e^{2t}}{D^2-4}-\frac{20e^t}{D^2-4}+\frac{17e^{0t}}{D^2-4}\right]$$

$$= \frac{1}{81}\left[t\cdot\frac{8e^{2t}}{2D}-\frac{20e^t}{-3}+\frac{17}{-4}\right]$$

$$y_p = \frac{1}{81}\left[t\cdot 2e^{2t}+\frac{20}{3}e^t-\frac{17}{4}\right]$$

Complete solution : $y = y_c+y_p$

Thus $y = c_1(3x+2)^2+\frac{c_2}{(3x+2)^2}+\frac{1}{81}\left[2\log(3x+2)(3x+2)^2+\frac{20}{3}(3x+2)-\frac{17}{4}\right]$

EXERCISES

Solve the following differential equations

1. $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y = 0$
2. $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y = (1+x)^2$
3. $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y = 10\left(x+\frac{1}{x}\right)$
4. $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y = \sin(\log x)$
5. $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-12y = x^2\log x$

6. $2xy'' + 3y' - \frac{y}{x} = 5 - \frac{\sin(\log x)}{x^2}$

7. $(1+x)^2 y'' + (1+x)y' + y = 4\cos\log(1+x)$

8. $(2x+1)^2 y'' - 2(2x+1)y' - 12y = x\log(2x+1)$

9. $(3x-2)^2 y'' - 3(3x-2)y' = 9(3x-2)\sin\log(3x-2)$

10. $(x+2)^2 y'' - (x+2)y' + y = 3x+4$

ANSWERS

1. $y = (c_1 + c_2\log x)x + c_3/x^2$

2. $y = (c_1 + c_2\log x)x^2 + \frac{x^2(\log x)^2}{2} + 2x + \frac{1}{4}$

3. $y = \frac{c_1}{x} + x\{c_2\cos(\log x) + c_3\sin(\log x)\} + 5x + \frac{2\log x}{x}$

4. $y = (c_1 + c_2\log x)x^2 + \frac{1}{25}\{3\sin(\log x) + 4\cos(\log x)\}$

5. $y = \frac{c_1}{x^4} + c_2 x^3 - \frac{x^2}{6}\left(\log x + \frac{5}{6}\right)$

6. $y = \frac{c_1}{x} + c_2\sqrt{x} + \frac{5x}{2} - \frac{1}{13x}\{3\cos(\log x) - 2\sin(\log x)\}$

7. $y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] + 2\log(1+x)\sin[\log(1+x)]$

8. $y = c_1 x^3 + \frac{c_2}{x} - \frac{\log(2x+1)}{32}(2x+1) + \frac{\log(2x+1)}{24} - \frac{1}{36}$

9. $y = c_1 + c_2(3x-2)^2 - \frac{1}{2}(3x-2)\sin[\log(3x-2)]$

10. $y = c_1\cos(\log x) + c_2\sin(\log x) + \frac{1}{4}\log x[\sin(\log x) - \log x\cos(\log x)]$

3.4 Series Solution of Differential Equation

We discuss the method of finding solution of a second order homogeneous differential equation in the form of convergent infinite power series.

Subsequently we extend / generalize this method which is referred to as generalized power series method or **Frobenius method**.

3.41 Power series solution of a second order ODE

Consider the DE in the form

$$P_0(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)\,y=0 \qquad \ldots(1)$$

where $P_0(x)$, $P_1(x)$ and $P_2(x)$ are polynomials in x with $\mathbf{P_0(x) \neq 0}$ **at** $\mathbf{x = 0}$.

The method is explained step wise.

- We assume the solution of (1) in the form of a power series,

$$y=\sum_{r=0}^{\infty} a_r x^r \qquad \ldots(2)$$

- Then, $\frac{dy}{dx}=y'=\sum_{0}^{\infty} a_r\, r x^{r-1}$ and $\frac{d^2y}{dx^2}=y''=\sum_{0}^{\infty} a_r\, r(r-1)\, x^{r-2}$

- We substitute these along with $y=\sum_{0}^{\infty} a_r\, x^r$ in (1) which results in an infinite series with various powers of x equal to zero.

[*It is evident that this will be satisfied only when the coefficients of the various powers of x are equal to zero.*]

- We equate the coefficients of various powers of x (*starting from the lowest power of* x) to zero. In general when the coefficient of x^r is equated to zero, we obtain a recurrence relation which will helps us to determine the constants $a_2, a_3, a_4, a_5, \cdots$ in terms of a_0 and a_1.

- We substitute the values of $a_2, a_3, a_4, \ldots$ in the expanded form of (2):

$$y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots$$

- Thus we get the power series solution of the ODE in the form,

$$y=a_0\,F(x)+a_1\,G(x)$$

where $F(x)$ and $G(x)$ are convergent infinite series in x.

Remarks :

1. *The method is adoptable for first order DEs also.*
2. *Sometimes we can recognize the functions $F(x)$ and $G(x)$ represented by convergent infinite series. This will help us to compare the series solution with that of analytical solution. We recall the following* ***standard convergent infinite series*** *in ascending powers of x.*

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ **2.** $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$

3. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$ **4.** $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

5. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ **6.** $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Note : *The first three of the worked problems that follows are presented with the intention of giving an insight to the series solution method by taking DEs having analytical solution. Various steps are also explained in detail.*

WORKED PROBLEMS

33. *Obtain the series solution of the equation* $\frac{dy}{dx} - 2xy = 0$

>> We have, $y' - 2xy = 0$ . . . (1)

[*Note that the coefficient of* $y' = 1 = P_0(x) \neq 0$ *at* $x = 0$]

Let $$y = \sum_{r=0}^{\infty} a_r x^r \qquad \ldots (2)$$

be the series solution of the given d.e.

$$\therefore \quad y' = \sum_{0}^{\infty} a_r r x^{r-1}$$

Now (i) becomes,

$$\sum_{0}^{\infty} a_r r x^{r-1} - 2x \sum_{0}^{\infty} a_r x^r = 0$$

ie., $$\sum_{0}^{\infty} a_r r x^{r-1} - 2 \sum_{0}^{\infty} a_r x^{r+1} = 0$$

We equate the coefficients of various powers of x to zero.

[*Note that on giving values for* $r = 0, 1, 2, 3, \cdots$ *the first summation has terms with powers of* $x : x^{-1}, x^0, x^1, x^2, \cdots$ *and the second summation has terms with powers* $x^1, x^2, \cdots$ *We independently equate the coefficient of* x^{-1} *and* x^0 *to zero first and subsequently equate the coefficient of* x^r *in general to zero*]

$$\text{Coeff. of } x^{-1} : a_0(0) = 0 \quad \Rightarrow a_0 \neq 0$$

$$\text{Coeff. of } x^0 : a_1(1) = 0 \quad \Rightarrow a_1 = 0$$

Now we shall equate the coefficient of x^r ($r \geq 1$) to zero.

ie., $\quad a_{r+1}(r+1) - 2a_{r-1} = 0$

or $\quad a_{r+1} = \dfrac{2a_{r-1}}{r+1} \; ; \; r \geq 1 \qquad \ldots (3)$

(*This is the recurrence relation which helps us to find* $a_2, a_3, a_4, \ldots$)

By putting $r = 1, 2, 3, 4, 5, \ldots$ in (3) we obtain,

$$a_2 = \frac{2a_0}{2} = a_0 \; ; \; a_3 = \frac{2a_1}{3} = 0, \text{since } a_1 = 0$$

$$a_4 = \frac{2a_2}{4} = \frac{1}{2}a_0 ; \; a_5 = \frac{2a_3}{5} = 0, \text{since } a_3 = 0$$

$$a_6 = \frac{2a_4}{6} = \frac{1}{3} \cdot \frac{1}{2} a_0 = \frac{a_0}{6} ; \; a_7 = \frac{2a_5}{7} = 0, \text{since } a_5 = 0 \text{ and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

ie., $\quad y = a_0 + a_2 x^2 + a_4 x^4 + \cdots, \text{since } a_1 = 0 = a_3 = a_5 = \cdots$

ie., $\quad y = a_0 \left[1 + x^2 + \dfrac{x^4}{2} + \dfrac{x^6}{6} + \cdots \right] \qquad \ldots (4)$

This is the required series solution of the given DE.

Remark : ***Comparison with the analytical solution***

We have , $\dfrac{dy}{dx} = 2xy$

or $\dfrac{dy}{y} = 2x\,dx$, *by separating the variables.*

$\Rightarrow \quad \int \frac{dy}{y} = \int 2x\,dx + c$

ie., $\quad \log_e y = x^2 + c \quad$ *or* $\quad y = e^{x^2+c} = e^c e^{x^2}$

Denoting $e^c = k,\ $ $y = ke^{x^2}$ ***is the analytical solution of (1).***

*The **series solution** obtained by us as in (4) can be written in the form*

$$y = a_0 \left[1 + \frac{x^2}{1!} + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots \right]$$

ie., $\quad y = a_0 e^{x^2}$ *[Refer series (1)] is same as the analytical solution obtained.*

---

34. *Obtain the series solution of the equation* $\frac{d^2y}{dx^2} + y = 0$

>> We have, $y'' + y = 0 \qquad \ldots(1)$

(Note that the coefficient of $y'' = 1 = P_0(x) \neq 0$ *at* $x = 0$*).*

Let $\quad y = \sum_{r=0}^{\infty} a_r x^r \qquad \ldots(2)$

be the series solution of (1).

$$\therefore \quad y' = \sum_0^{\infty} a_r r x^{r-1},\quad y'' = \sum_0^{\infty} a_r r(r-1) x^{r-2}$$

Now (1) becomes,

$$\sum_0^{\infty} a_r r(r-1) x^{r-2} + \sum_0^{\infty} a_r x^r = 0$$

We equate the coefficients of various powers of x to zero.

[Note that the first summation has terms with powers of $x : x^{-2}, x^{-1}, x^0, x^1, \ldots$ *and the second summation has terms with powers of* $x : x^0, x^1, x^2, \ldots$*]*

Coeff. of x^{-2}: $a_0(0)(-1) = 0 \quad \Rightarrow a_0 \neq 0$

Coeff. of x^{-1}: $a_1(1)(0) = 0 \quad \Rightarrow a_1 \neq 0$

Now we shall equate the coefficient of x^r $(r \geq 0)$ to zero.

ie., $a_{r+2}(r+2)(r+1)+a_r = 0$

or $$a_{r+2} = \frac{-a_r}{(r+2)(r+1)} \quad (r \geq 0) \qquad \ldots (3)$$

(This is the recurrerence relation which helps us to find $a_2, a_3, a_4, \ldots$)

By putting $r = 0, 1, 2, 3, 4, \ldots$ in (3) we obtain,

$$a_2 = -\frac{a_0}{2} \quad ; \quad a_3 = -\frac{a_1}{6}$$

$$a_4 = \frac{-a_2}{12} = \frac{a_0}{24} \quad ; \quad a_5 = \frac{-a_3}{20} = \frac{a_1}{120}$$

$$a_6 = -\frac{a_4}{30} = \frac{-a_0}{720} \quad ; \quad a_7 = -\frac{a_5}{42} = \frac{-a_1}{5040} \quad \text{and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

ie., $$y = a_0 \left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right] + a_1 \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right]$$

Thus $$y = a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right] + a_1 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right]$$

is the required series solution of the given d.e.

Remark : ***Comparison with the analytical solution.***

We have, $(D^2 + 1)\,y = 0$, *where* $D = \frac{d}{dx}$

Auxilary equation is $m^2 + 1 = 0$ *and* $m = \pm i$ *are its roots.*

The ***analytical solution*** *is given by*

$$y = c_1 \cos x + c_2 \sin x$$

The ***series solution*** *obtained by us can be represented as*

$$y = a_0 \cos x + a_1 \sin x$$ *[Refer standard convergent series (5) and (6)]*

This is same as the analytical solution.

---

35. *Obtain the power series solution of the equation* $\frac{d^2 y}{dx^2} - y = 0$

>> This example is very much similar to the previous example. Proceeding on the same lines we obtain, $a_0 \neq 0, a_1 \neq 0$ and the recurrence relation is as follows.

$$a_{r+2} = \frac{a_r}{(r+2)(r+1)} \quad (r \geq 0)$$

By putting $r = 0, 1, 2, 3, 4, \ldots$ we obtain,

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{6}, \quad a_4 = \frac{a_0}{24}, \quad a_5 = \frac{a_1}{120}, \quad a_6 = \frac{a_0}{720}, \ldots$$

Substituting these values in the expanded form of $y = \sum_{r=0}^{\infty} a_r x^r$ we obtain,

$$y = a_0\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right] + a_1\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]$$

This is the required series soution of the given d.e.

Remark : ***Comparison with the analytical solution.***

We have $(D^2 - 1)y = 0$, *where* $D = \frac{d}{dx}$

Auxilary equation is $m^2 - 1 = 0$ *and hence* $m = \pm 1$ *are its roots.*

The ***analytical soution*** *is given by* $y = c_1 e^x + c_2 e^{-x}$

The ***series solution*** *obtained by us can be represented as*

$$y = a_0 \cosh x + a_1 \sinh x \quad \text{[Refer standard series (3) and (4)]}$$

ie., $$y = a_0\left[\frac{e^x + e^{-x}}{2}\right] + a_1\left[\frac{e^x - e^{-x}}{2}\right]$$

or $$y = \left[\frac{a_0 + a_1}{2}\right] e^x + \left[\frac{a_0 - a_1}{2}\right] e^{-x}$$

ie., $y = c_1 e^x + c_2 e^{-x}$, where $c_1 = \frac{a_0 + a_1}{2}$ and $c_2 = \frac{a_0 - a_1}{2}$

This is same as the analytical solution obtained by us.

---

36. *Solve* $\dfrac{d^2y}{dx^2}+xy=0$ *by obtaining the solution in the form of series.*

>> We have $y''+xy=0$ $\quad\ldots(1)$

The coefficient of $y''=1=P_0(x)\neq 0$ at $x=0$.

Let $$y=\sum_{r=0}^{\infty}a_r x^r \quad\ldots(2)$$

be the series solution of (1).

$$\therefore\quad y'=\sum_{0}^{\infty}a_r r x^{r-1},\quad y''=\sum_{0}^{\infty}a_r r(r-1)x^{r-2}$$

Now (1) becomes,

$$\sum_{0}^{\infty}a_r r(r-1)x^{r-2}+\sum_{0}^{\infty}a_r x^{r+1}=0$$

We equate the coefficients of various powers of x to zero.

We first equate the cofficients of x^{-2}, x^{-1} and x^0 available only in the first summation to zero.

Coeff. of x^{-2} : $a_0(0)(-1)=0 \Rightarrow a_0\neq 0$

Coeff. of x^{-1} : $a_1(1)(0)=0 \Rightarrow a_1\neq 0$

Coeff. of x^0 : $a_2(2)(1)=0$ or $2a_2=0\Rightarrow a_2=0$

Now we shall equate the coefficient of x^r $(r\geq 1)$ to zero.

ie., $\quad a_{r+2}(r+2)(r+1)+a_{r-1}=0$

or $$a_{r+2}=\frac{-a_{r-1}}{(r+2)(r+1)}\quad (r\geq 1)\quad\ldots(3)$$

By putting $r=1,2,3,4,\ldots$ in (3) we obtain,

$$a_3=\frac{-a_0}{6};\quad a_4=\frac{-a_1}{12};\quad a_5=\frac{-a_2}{20}=0$$

$$a_6=-\frac{a_3}{30}=\frac{a_0}{180};\quad a_7=\frac{-a_4}{42}=\frac{a_1}{504};\quad a_8=0=a_{11}=a_{14}\ldots$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

Thus, $y = a_0 \left[1 - \frac{x^3}{6} + \frac{x^6}{180} - \cdots \right] + a_1 \left[x - \frac{x^4}{12} + \frac{x^7}{504} - \cdots \right]$

is the required solution in the form of series.

---

37. *Obtain the series soution of the equation* $\frac{d^2 y}{dx^2} + x^2 y = 0$

>> We have $y'' + x^2 y = 0$ ... (1)

The coefficient of $y'' = 1 = P_0(x) \neq 0$ at $x = 0$.

Let $y = \sum_{r=0}^{\infty} a_r x^r$ ... (2)

be the series solution of (1).

$$\therefore \quad y' = \sum_0^{\infty} a_r r x^{r-1}, \quad y'' = \sum_0^{\infty} a_r r(r-1) x^{r-2}$$

Now (1) becomes,

$$\sum_0^{\infty} a_r r(r-1) x^{r-2} + \sum_0^{\infty} a_r x^{r+2} = 0$$

We equate the coefficients of various powers of x to zero.

We first equate the coefficients of x^{-2}, x^{-1}, x^0, x^1 available only in the first summation to zero.

Coeff. of x^{-2} : $a_0(0)(-1) = 0 \Rightarrow a_0 \neq 0$

Coeff. of x^{-1} : $a_1(1)(0) = 0 \Rightarrow a_1 \neq 0$

Coeff. of x^0 : $a_2(2)(1) = 0$ or $2a_2 = 0 \Rightarrow a_2 = 0$

Coeff. of x^1 : $a_3(3)(2) = 0$ or $6a_3 = 0 \Rightarrow a_3 = 0$

Now we shall equate the coefficient of x^r $(r \geq 2)$ to zero.

ie., $a_{r+2}(r+2)(r+1)+a_{r-2}=0$

or $$a_{r+2}=\frac{-a_{r-2}}{(r+2)(r+1)} \quad (r \geq 2) \qquad \ldots(3)$$

By putting $r=2,3,4,5\cdots$ in (3) we obtain,

$$a_4=-\frac{a_0}{4\cdot 3};\ a_5=\frac{-a_1}{5\cdot 4};\ a_6=\frac{-a_2}{6\cdot 5}=0;\ a_7=\frac{-a_3}{7\cdot 6}=0$$

$$a_8=-\frac{a_4}{8\cdot 7}=\frac{a_0}{8\cdot 7\cdot 4\cdot 3};\ a_9=-\frac{a_5}{9\cdot 8}=\frac{a_1}{9\cdot 8\cdot 5\cdot 4};$$

$a_{10}=0$; $a_{11}=0$ and so-on.

We substitute these values in the expanded form of (2) :

$$y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots$$

Thus $$y=a_0\left[1-\frac{x^4}{4\cdot 3}+\frac{x^8}{8\cdot 7\cdot 4\cdot 3}-\cdots\right]+a_1\left[x-\frac{x^5}{5\cdot 4}+\frac{x^9}{9\cdot 8\cdot 5\cdot 4}-\cdots\right]$$

is the required series solution.

---

38. *Solve* $y''+xy'+y=0$ *in series.*

>> $y''+xy'+y=0 \qquad \ldots(1)$

The coefficient of $y''=1=P_0(x)\neq 0$ at $x=0$.

Let $$y=\sum_{r=0}^{\infty} a_r x^r \qquad \ldots(2)$$

be the series solution of (1).

$$\therefore \quad y'=\sum_{0}^{\infty} a_r\, r\, x^{r-1},\quad y''=\sum_{0}^{\infty} a_r\, r(r-1)\, x^{r-2}$$

Now (1) becomes,

$$\sum_{0}^{\infty} a_r\, r(r-1)\, x^{r-2}+\sum_{0}^{\infty} a_r r x^r+\sum_{0}^{\infty} a_r x^r=0$$

We equate the coefficients of various powers of x to zero.

We first equate the coefficients of x^{-2}, x^{-1} available only in the first summation to zero.

Coeff. of x^{-2} : $a_0\,(0)\,(-1) = 0 \quad \Rightarrow \quad a_0 \neq 0$

Coeff. of x^{-1} : $a_1\,(1)\,(0) = 0 \quad \Rightarrow \quad a_1 \neq 0$

Now we shall equate the coefficient of x^r $(r \geq 0)$ to zero.

ie., $a_{r+2}\,(r+2)\,(r+1) + a_r r + a_r = 0$

ie., $a_{r+2}\,(r+2)\,(r+1) + (r+1)\,a_r = 0$

ie., $a_{r+2}\,(r+2) + a_r = 0$

or $$a_{r+2} = \frac{-a_r}{r+2} \quad (r \geq 0) \qquad \ldots (3)$$

By putting $r = 0, 1, 2, 3, 4, \cdots$ in (3) we obtain,

$$a_2 = -\frac{a_0}{2}\,;\; a_3 = -\frac{a_1}{3}\,;\; a_4 = -\frac{a_2}{4} = \frac{a_0}{2 \cdot 4}\,;\; a_5 = -\frac{a_3}{5} = \frac{a_1}{3 \cdot 5}$$

$$a_6 = -\frac{a_4}{6} = \frac{-a_0}{2 \cdot 4 \cdot 6}\,;\; a_7 = -\frac{a_5}{7} = \frac{-a_1}{3 \cdot 5 \cdot 7} \text{ and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Thus $$y = a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots\right] + a_1\left[x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} - \frac{x^7}{3 \cdot 5 \cdot 7} + \cdots\right]$$

is the required series solution.

---

39. *Solve* $y'' - xy' + y = 0$ *in series.*

\>> This example is very much similar to the previous one. Proceeding on the same lines we do obtain $a_0 \neq 0$, $a_1 \neq 0$. The coefficient of x^r $(r \geq 0)$ when equated to zero will result in,

$$a_{r+2}\,(r+2)(r+1) - a_r r + a_r = 0$$

ie., $a_{r+2}\,(r+2)(r+1) = a_r(r-1)$

or $$a_{r+2} = \frac{(r-1)\,a_r}{(r+2)(r+1)} \quad (r \geq 0)$$

By putting $r = 0, 1, 2, 3, 4, \ldots$ we obtain,

$a_2 = -\frac{a_0}{2}$; $a_3 = 0$; $a_4 = \frac{a_2}{4 \cdot 3} = \frac{-a_0}{2 \cdot 3 \cdot 4}$; $a_5 = 0$; $a_6 = \frac{3a_4}{6 \cdot 5} = \frac{-3a_0}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$;

$a_7 = 0$ and so-on.

We substitute these values in the series,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Thus $y = a_0 \left[1 - \frac{x^2}{2} - \frac{x^4}{24} - \frac{x^6}{240} - \cdots \right] + a_1 x$ is the required series solution.

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40. ***Develop the series solution of the equation*** $y'' + x y' + (x^2 + 2) y = 0$

>> $$y'' + x y' + (x^2 + 2) y = 0 \qquad \ldots (1)$$

The coefficient of $y'' = 1 = P_0(x) \neq 0$ at $x = 0$.

Let $$y = \sum_{r=0}^{\infty} a_r x^r \qquad \ldots (2)$$

be the series solution of (1).

$$\therefore \quad y' = \sum_0^{\infty} a_r r x^{r-1}, \quad y'' = \sum_0^{\infty} a_r r(r-1) x^{r-2}$$

Now (1) becomes,

$$\sum_0^{\infty} a_r r(r-1) x^{r-2} + \sum_0^{\infty} a_r r x^r + \sum_0^{\infty} a_r x^{r+2} + 2 \sum_0^{\infty} a_r x^r = 0$$

We equate the coefficients of various powers of x to zero. We first equate the coefficients of x^{-2}, x^{-1}, x^0, x^1 (*available in various summations except the third one*) to zero.

Coeff. of x^{-2} : $a_0(0)(-1) = 0 \Rightarrow a_0 \neq 0$

Coeff. of x^{-1} : $a_1(1)(0) = 0 \Rightarrow a_1 \neq 0$

Coeff. of x^0 : $a_2(2)(1) + a_0(0) + 2a_0 = 0$

ie., $2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0$

Coeff. of x^1 : $a_3(3)(2) + a_1(1) + 2a_1 = 0$

ie., $6a_3 + 3a_1 = 0 \Rightarrow a_3 = -a_1/2$

Now we shall equate the coefficient of x^r $(r \geq 2)$ to zero.

ie., $a_{r+2}(r+2)(r+1)+a_r r + a_{r-2} + 2a_r = 0$

ie., $$a_{r+2} = \frac{-[a_{r-2} + (r+2)a_r]}{(r+2)(r+1)}$$

or $$a_{r+2} = \frac{-a_{r-2}}{(r+2)(r+1)} - \frac{a_r}{(r+1)} \quad (r \geq 2)$$

By putting $r = 2, 3, 4, 5, \ldots$ in (3) we obtain,

$$a_4 = \frac{-a_0}{12} - \frac{a_2}{3} = -\frac{a_0}{12} + \frac{a_0}{3} = \frac{a_0}{4}$$

$$a_5 = \frac{-a_1}{20} - \frac{a_3}{4} = -\frac{a_1}{20} + \frac{a_1}{8} = \frac{3a_1}{40}$$

$$a_6 = \frac{-a_2}{30} - \frac{a_4}{5} = \frac{a_0}{30} - \frac{a_0}{20} = -\frac{a_0}{60} \text{ and so on.}$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Thus $$y = a_0\left[1 - x^2 + \frac{x^4}{4} - \frac{x^6}{60} + \cdots\right] + a_1\left[x - \frac{x^3}{2} + \frac{3x^5}{40} - \cdots\right]$$

is the required series solution.

---

41. *Develop the series solution of the equation* $(1+x^2)\frac{d^2 y}{dx^2} + x\frac{dy}{dx} - y = 0$

>> We have $(1+x^2)y'' + xy' - y = 0$ ... (1)

The coefficient of $y'' = 1 + x^2 = P_0(x)$ and we have at $x = 0$, $P_0(x) = 1 \neq 0$

Let $$y = \sum_{r=0}^{\infty} a_r x^r \quad \ldots (2)$$

be the series soluiton of (1).

$$\therefore \quad y' = \sum_{0}^{\infty} a_r r x^{r-1}, \quad y'' = \sum_{0}^{\infty} a_r r(r-1)x^{r-2}$$

Now (1) becomes,

$$\sum_0^{\infty} a_r r(r-1)x^{r-2} + \sum_0^{\infty} a_r r(r-1)x^r + \sum_0^{\infty} a_r r x^r - \sum_0^{\infty} a_r x^r = 0$$

We equate the coefficients of various powers of x to zero.

We first equate the coefficients of x^{-2}, x^{-1} available only in the first summation to zero.

Coeff. of x^{-2} : $a_0(0)(-1) = 0 \Rightarrow a_0 \neq 0$

Coeff. of x^{-1} : $a_1(1)(0) = 0 \Rightarrow a_1 \neq 0$

Now we shall equate the coefficient of x^r $(r \geq 0)$ to zero.

ie., $a_{r+2}(r+2)(r+1) + a_r r(r-1) + a_r r - a_r = 0$

ie., $a_{r+2}(r+2)(r+1) + a_r(r^2 - r + r - 1) = 0$

ie., $a_{r+2}(r+2)(r+1) + a_r(r^2 - 1) = 0$

ie., $a_{r+2}(r+2)(r+1) + a_r(r+1)(r-1) = 0$

or $a_{r+2} = -\dfrac{(r-1)a_r}{(r+2)}$ $(r \geq 0)$

By putting $r = 0, 1, 2, 3, \ldots$ in (3) we obtain,

$a_2 = \dfrac{a_0}{2}$; $a_3 = 0$; $a_4 = -\dfrac{a_2}{4} = -\dfrac{a_0}{8}$; $a_5 = \dfrac{-2a_3}{5} = 0$; $a_6 = \dfrac{-3a_4}{6} = \dfrac{a_0}{16}$;
$a_7 = 0$; and so-on.

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Thus $y = a_0\left[1 + \dfrac{x^2}{2} - \dfrac{x^4}{8} + \dfrac{x^6}{16} - \cdots\right] + a_1 x$ is the required series solution.

42. *Solve in series the equation* $(x-1)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0$ *subject to the conditions* $y(0)=2$ *and* $y'(0)=-1$

>> We have, $(x-1)y''+xy'+y=0$ ... (1)

The coefficient of $y''=(x-1)=P_0(x)$ and at $x=0$, $P_0(x)=-1\neq 0$.

Let $$y=\sum_{r=0}^{\infty} a_r x^r \qquad \ldots (2)$$

be the series solution of (1).

$$\therefore \quad y'=\sum_{0}^{\infty} a_r r x^{r-1}, \quad y''=\sum_{0}^{\infty} a_r r(r-1)x^{r-2}$$

Now (1) becomes,

$$\sum_{0}^{\infty} a_r r(r-1)x^{r-1}-\sum_{0}^{\infty} a_r r(r-1)x^{r-2}+\sum_{0}^{\infty} a_r r x^r+\sum_{0}^{\infty} a_r x^r=0$$

We equate the coefficients of various powers of x to zero.

We first equate the coefficients of x^{-2}, x^{-1} to zero.

Coeff. of x^{-2} : $-a_0(0)(1)=0 \Rightarrow a_0\neq 0$

Coeff. of x^{-1} : $a_0(0)(-1)-a_1(1)(0)=0 \Rightarrow a_1\neq 0$

Now we shall equate the coefficient of x^r $(r\geq 0)$ to zero.

ie., $a_{r+1}(r+1)r-a_{r+2}(r+2)(r+1)+a_r r+a_r=0$

ie., $a_{r+1}(r+1)r-a_{r+2}(r+2)(r+1)+a_r(r+1)=0$

ie., $a_{r+1}r-a_{r+2}(r+2)+a_r=0$

or $$a_{r+2}=\frac{r\,a_{r+1}+a_r}{r+2} \quad (r\geq 0) \qquad \ldots (3)$$

By putting $r=0,1,2,3,\ldots$ in (3) we obtain,

$$a_2=\frac{a_0}{2} \; ; \; a_3=\frac{a_2+a_1}{3}=\frac{a_0/2+a_1}{3}=\frac{a_0}{6}+\frac{a_1}{3}$$

$$a_4 = \frac{2a_3 + a_2}{4} = \frac{a_0/3 + 2a_1/3 + a_0/2}{4} = \frac{5a_0}{24} + \frac{a_1}{6}$$

$$a_5 = \frac{3a_4 + a_3}{5} = \frac{5a_0/8 + a_1/2 + a_0/6 + a_1/3}{5} = \frac{19a_0}{120} + \frac{a_1}{6} \text{ and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

ie., $$y = a_0\left[1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{5x^4}{24} + \frac{19x^5}{120} + \cdots\right] + a_1\left[x + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{6} + \cdots\right] \quad \ldots (4)$$

This is the general solution of (1) in series.

To apply the given initial conditions, we differentiate (4) w.r.t. x.

$$\therefore \quad y' = a_0\left[x + \frac{x^2}{2} + \frac{5x^3}{6} + \cdots\right] + a_1\left[1 + x^2 + \frac{2x^3}{3} + \cdots\right] \quad \ldots (5)$$

Using the conditions, $y = 2$ and $y' = -1$ and $x = 0$, (4) and (5) respectively becomes,

$$2 = a_0 \text{ and } -1 = a_1$$

Hence (4) becomes,

$$y = 2\left[1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{5x^4}{24} + \frac{19x^5}{120} + \cdots\right] - \left[x + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{6} \cdots\right]$$

Thus $y = 2 - x + x^2 + \dfrac{x^4}{4} + \dfrac{3x^5}{20} + \cdots$ **is the required particular solution in series.**

3.42 Generalized Power Series Method or Frobenius Method

Consider a second order DE in the form

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \quad \ldots (1)$$

where $P_0(x), P_1(x), P_2(x)$ are polynomials in x with $P_0(x) = 0$ at $x = 0$

The method is explained stepwise.

➲ We assume the series solution of (1) in the form

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \ldots (2)$$

where $k, a_0, a_1, a_2, \ldots$ are all constants and $a_0 \neq 0$

- Then $\frac{dy}{dx} = y' = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$

$$\frac{d^2 y}{dx^2} = y'' = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

- We substitute these along with $y = \sum_{r=0}^{\infty} a_r x^{k+r}$ in (1) which results in an infinite series with various powers of x equal to zero.
- We equate the coefficient of the lowest degree term in x to zero. This will give us a quadratic equation in k known as the ***indicial equation.*** Let k_1 and k_2 be the roots of this equation.
- We need to equate the coefficients of various other powers of x also to zero. In general when we equate the coefficient of x^{k+r} to zero, we obtain a recurrence relation which helps to determine the constants $a_1, a_2, a_3, a_4, \ldots$ in terms of a_0 only.
- We substitute the values of $a_1, a_2 \cdot a_3, \ldots$ in the expanded form of (2) given by

$$y = x^k (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

- Hence we obtain $y = a_0 x^k F(x)$ where $F(x)$ is an infinite series.
- We suppose that k_1, k_2 are real, distinct and do not differ by an integer, that is $k_1 - k_2 \neq 0, 1, 2, 3, \ldots$
- Then $y_1 = a_0 x^{k_1} F(x)$ & $y_2 = a_0 x^{k_2} F(x)$ are two independent solutions of (1).
- Thus $y = A y_1 + B y_2$ constitutes the general / complete solution of (1) in series, where A and B are arbitrary constants.

Remark : *It should be noted that the roots k_1, k_2 of the indicial equation can also be (a) real, distinct, differing by a non zero integer, that is $k_1 - k_2 = 1, 2, 3, \ldots$ (b) coincident, that is $k_1 = k_2 = c$ (say)*

The complete / general solution in the case (a) will be

$$y = A [y]_{k = k_1} + B \left[\frac{\partial y}{\partial k}\right]_{k = k_1} \quad \textit{(A, B are arbitrary constants)}$$

where $k_1 < k_2$ and $k = k_2$ will not result in a new independent solution.

The complete / general solution in the case (b) will be

$$y = A\,[\,y\,]_{k=c} + B\left[\frac{\partial y}{\partial k}\right]_{k=c}$$

Note : *Problems on these two cases are not presented as the focus of attention is mainly on the power series method and Frobenius method.*

WORKED PROBLEMS

43. *Solve by Frobenius method the equation* $4x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$

>> We have $4xy'' + 2y' + y = 0$ ... (1)

The coefficient of $y'' = 4x = P_0(x)$ and $P_0(x) = 0$ at $x = 0$.

Let $\quad y = \sum_{r=0}^{\infty} a_r x^{k+r}$ ... (2)

be the series solution of (1).

$$\therefore \quad y' = \sum_0^{\infty} a_r(k+r)x^{k+r-1}, \quad y'' = \sum_0^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$$

Now (1) becomes,

$$4\sum_0^{\infty} a_r(k+r)(k+r-1)x^{k+r-1} + 2\sum_0^{\infty} a_r(k+r)x^{k+r-1} + \sum_0^{\infty} a_r x^{k+r} = 0$$

We first equate the coefficient of the lowest degree term in x to zero.

On equating the coefficient of x^{k-1} to zero we have,

$4a_0 k(k-1) + 2a_0 k = 0$, which is the indicial equation.

ie., $\quad 2a_0 k(2k-2+1) = 0 \quad$ or $\quad a_0 k(2k-1) = 0$

Since $a_0 \neq 0$ we have $k = 0$ and $k = 1/2$

[*Note that these roots donot differ by an integer.*]

Next, we shall equate the coefficient of x^{k+r} $(r \geq 0)$ to zero.

ie., $\quad 4\,a_{r+1}(k+r+1)(k+r) + 2a_{r+1}(k+r+1) + a_r = 0$

ie., $\quad 2a_{r+1}(k+r+1)[2k+2r+1] + a_r = 0$

or $$a_{r+1} = \frac{-a_r}{2(k+r+1)(2k+2r+1)} \quad (r \geq 0) \qquad \ldots (3)$$

Case-(1) : Let $k = 0$

$$\therefore \quad a_{r+1} = \frac{-a_r}{2(r+1)(2r+1)} \quad (r \geq 0)$$

Putting $r = 0, 1, 2, 3, \ldots$ in this relation we obtain

$$a_1 = -\frac{a_0}{2} \; ; \; a_2 = -\frac{a_1}{12} = \frac{a_0}{24} \; ; \; a_3 = -\frac{a_2}{30} = -\frac{a_0}{720} \text{ and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = x^k (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

ie., $$y = a_0 \, x^k \left[1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \cdots \right]$$

The series solution obtained by putting $k = 0$ be denoted by y_1 .

$$\therefore \quad y_1 = a_0 \left[1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \cdots \right] \qquad \ldots (4)$$

Case-(2) : Let $k = 1/2$ and hence (3) assumes the form

$$a_{r+1} = \frac{-a_r}{2(r+3/2)(2r+2)} = \frac{-a_r}{(2r+3) \cdot 2(r+1)}$$

ie., $$a_{r+1} = \frac{-a_r}{2(r+1)(2r+3)} \quad (r \geq 0)$$

Putting $r = 0, 1, 2, 3$ in this relation we obtain.

$$a_1 = -\frac{a_0}{6} \; ; \; a_2 = -\frac{a_1}{20} = \frac{a_0}{120} \; ; \; a_3 = -\frac{a_2}{42} = -\frac{a_0}{5040} \text{ and so-on.}$$

We again substitute these values in the expanded form of (2) :

$$y = x^k (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

ie., $$y = a_0 \, x^k \left[1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \cdots \right]$$

The series solution obtained by putting $k = 1/2$ be denoted by y_2 .

$$\therefore \quad y_2 = a_0 x^{1/2}\left[1-\frac{x}{6}+\frac{x^2}{120}-\frac{x^3}{5040}+\cdots\right] \qquad \ldots(5)$$

The complete solution of (1) is given by

$$y = Ay_1 + By_2$$

ie.,
$$y = Aa_0\left[1-\frac{x}{2}+\frac{x^2}{24}-\frac{x^3}{720}+\cdots\right]+Ba_0\sqrt{x}\left[1-\frac{x}{6}+\frac{x^2}{120}-\frac{x^3}{5040}+\cdots\right]$$

Let us denote $c_1 = Aa_0$ and $c_2 = Ba_0$

Thus $y = c_1\left[1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+\cdots\right]+c_2\sqrt{x}\left[1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+\cdots\right]$

is the required series solution.

Note : *The series solution obtained can be put in the following form :*

$$y = c_1\left[1-\frac{(\sqrt{x})^2}{2!}+\frac{(\sqrt{x})^4}{4!}-\frac{(\sqrt{x})^6}{6!}+\cdots\right]+c_2\left[\sqrt{x}-\frac{(\sqrt{x})^3}{3!}+\frac{(\sqrt{x})^5}{5!}-\frac{(\sqrt{x})^7}{7!}+\cdots\right]$$

ie., $\quad \boldsymbol{y = c_1 \cos(\sqrt{x}) + c_2 \sin(\sqrt{x})}$ [*Refer standard series (5) and (6) given earlier*]

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44. *Solve in series the equation* $2x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+(1-x^2)y = 0$

We have, $2x^2y''-xy'+(1-x^2)y = 0 \qquad \ldots(1)$

The coefficient of $y'' = 2x^2 = P_0(x)$ and $P_0(x) = 0$ at $x = 0$.

Let $\quad y = \sum_{r=0}^{\infty} a_r x^{k+r} \qquad \ldots(2)$

be the seris soution of (1).

$$\therefore \quad y' = \sum_0^{\infty} a_r(k+r)x^{k+r-1}, \quad y'' = \sum_0^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$$

Now (1) becomes,

$$2\sum_{0}^{\infty} a_r (k+r)(k+r-1)x^{k+r} - \sum_{0}^{\infty} a_r (k+r)x^{k+r} + \sum_{0}^{\infty} a_r x^{k+r} - \sum_{0}^{\infty} a_r x^{k+r+2} = 0$$

We shall first equate the coefficient of the lowest degree term in x, that is x^k to zero.

ie., $2a_0 k(k-1) - a_0 k + a_0 = 0$ or $a_0(2k^2 - 2k - k + 1) = 0$

ie., $a_0(2k^2 - 3k + 1) = 0$ or $a_0(k-1)(2k-1) = 0$

Since $a_0 \neq 0$, we have $k = 1$ and $k = 1/2$.

(The roots donot differ by an integer.)

Next, we shall equate the coefficient of x^{k+1} to zero.

ie., $2a_1(k+1)k - a_1(k+1) + a_1 = 0$

ie., $a_1(2k^2 + k) = 0$ or $a_1 k(2k+1) = 0 \Rightarrow a_1 = 0, k = 0, k = -1/2$

Rejecting $k = 0$ and $-1/2$ *(since we already have $k = 1$ and $1/2$) we must have* $a_1 = 0$.

Now, we shall equate the coefficient of x^{k+r} $(r \geq 2)$ to zero.

ie., $2a_r(k+r)(k+r-1) - a_r(k+r) + a_r - a_{r-2} = 0$

ie., $2a_r(k+r)(k+r-1) - a_r(k+r-1) - a_{r-2} = 0$

ie., $a_r(k+r-1)[2k+2r-1] = a_{r-2}$

or $$a_r = \frac{a_{r-2}}{(k+r-1)(2k+2r-1)} \quad (r \geq 2) \qquad \ldots(3)$$

Case - (1) : Let $k = 1$

$$\therefore \quad a_r = \frac{a_{r-2}}{r(2r+1)} \quad (r \geq 2)$$

Putting $r = 2, 3, 4, \ldots$ in this relation we obtain,

$$a_2 = \frac{a_0}{2 \cdot 5}\ ;\ a_3 = \frac{a_1}{3 \cdot 7} = 0\ ;\ a_4 = \frac{a_2}{4 \cdot 9} = \frac{a_0}{2 \cdot 4 \cdot 5 \cdot 9}\ ;\ a_5 = 0$$

$$a_6 = \frac{a_4}{6 \cdot 13} = \frac{a_0}{2 \cdot 4 \cdot 6 \cdot 5 \cdot 9 \cdot 13} \text{ and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = x^k (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

ie.,

$$y = a_0 x^k \left[1 + \frac{x^2}{2.5} + \frac{x^4}{2 \cdot 4 \cdot 5 \cdot 9} + \frac{x^6}{2 \cdot 4 \cdot 6 \cdot 5 \cdot 9 \cdot 13} + \cdots \right]$$

Putting $k = 1$, we have

$$y_1 = a_0 x \left[1 + \frac{x^2}{2 \cdot 5} + \frac{x^4}{2 \cdot 4 \cdot 5 \cdot 9} + \frac{x^6}{2 \cdot 4 \cdot 6 \cdot 5 \cdot 9 \cdot 13} + \cdots \right] \quad \ldots (4)$$

Case-(2) : Let $k = 1/2$ and hence (3) assumes the form

$$a_r = \frac{a_{r-2}}{(r - 1/2)\, 2r} = \frac{a_{r-2}}{r(2r-1)} \quad (r \geq 2)$$

Putting $r = 2, 3, 4, \ldots$ we obtain

$$a_2 = \frac{a_0}{2 \cdot 3}, \quad a_3 = 0 = a_5 = \cdots$$

$$a_4 = \frac{a_2}{4 \cdot 7} = \frac{a_0}{2 \cdot 4 \cdot 3 \cdot 7}; \quad a_6 = \frac{a_4}{6 \cdot 11} = \frac{a_0}{2 \cdot 4 \cdot 6 \cdot 3 \cdot 7 \cdot 11} \text{ and so-on}$$

We substitute these values in the expanded form of (2) to obtain,

$$y = a_0 x^k \left[1 + \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 4 \cdot 3 \cdot 7} + \frac{x^6}{2 \cdot 4 \cdot 6 \cdot 3 \cdot 7 \cdot 11} + \cdots \right]$$

Putting $k = 1/2$ we have,

$$y_2 = a_0 \sqrt{x} \left[1 + \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 4 \cdot 3 \cdot 7} + \frac{x^6}{2 \cdot 4 \cdot 6 \cdot 3 \cdot 7 \cdot 11} + \cdots \right] \quad \ldots (5)$$

The complete solution of (1) is given by

$$y = Ay_1 + By_2$$

Let us denote $Aa_0 = c_1$ and $Ba_0 = c_2$

Thus the required series solution is given by

$$y = c_1 x \left[1 + \frac{x^2}{2 \cdot 5} + \frac{x^4}{2 \cdot 4 \cdot 5 \cdot 9} + \frac{x^6}{2 \cdot 4 \cdot 6 \cdot 5 \cdot 9 \cdot 13} + \cdots \right]$$

$$+ c_2 \sqrt{x} \left[1 + \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 4 \cdot 3 \cdot 7} + \frac{x^6}{2 \cdot 4 \cdot 6 \cdot 3 \cdot 7 \cdot 11} + \cdots \right]$$

---

45. *Obtain the series solution of the equation* $4xy'' + 2(1-x)y' - y = 0$

>> $4xy'' + (2-2x)y' - y = 0$ ... (1)

The coefficient of $y'' = 4x = P_0(x)$ and $P_0(x) = 0$ at $x = 0$.

Let $y = \sum_{r=0}^{\infty} a_r x^{k+r}$ ... (2)

be the series solution of (1).

$$\therefore \quad y' = \sum_0^{\infty} a_r (k+r) x^{k+r-1}, \quad y'' = \sum_0^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

Now (1) becomes,

$$4 \sum_0^{\infty} a_r (k+r)(k+r-1) x^{k+r-1} + 2 \sum_0^{\infty} a_r (k+r) x^{k+r-1}$$

$$- 2 \sum_0^{\infty} a_r (k+r) x^{k+r} - \sum_0^{\infty} a_r x^{k+r} = 0$$

We shall equate the coefficient of the lowest degreee term in x, that is x^{k-1} to zero.

ie., $4a_0 k(k-1) + 2a_0 k = 0$

ie., $2a_0 k(2k-2+1) = 0$ or $a_0 k(2k-1) = 0$

Since $a_0 \neq 0$ we have $k = 0$ and $k = 1/2$

Next, we shall equate the coefficient of x^{k+r} $(r \geq 0)$ to zero.

ie., $4a_{r+1}(k+r+1)(k+r) + 2a_{r+1}(k+r+1) - 2a_r(k+r) - a_r = 0$

ie., $2a_{r+1}(k+r+1)[2(k+r)+1] - a_r[2(k+r)+1] = 0$

ie., $2a_{r+1}(k+r+1)-a_r = 0$

or $$a_{r+1} = \frac{a_r}{2(k+r+1)} \quad (r \geq 0) \qquad \ldots (3)$$

Case-(1) : Let $k = 0$

$$\therefore \quad a_{r+1} = \frac{a_r}{2(r+1)} \quad (r \geq 0)$$

Putting $r = 0, 1, 2, 3, 4, \ldots$ we obtain,

$$a_1 = \frac{a_0}{2};\ a_2 = \frac{a_1}{4} = \frac{a_0}{8};\ a_3 = \frac{a_2}{2 \cdot 3} = \frac{a_0}{48};\ a_4 = \frac{a_3}{2 \cdot 4} = \frac{a_0}{384} \text{ and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = x^k(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

ie., $$y = a_0 x^k \left[1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \cdots \right]$$

Putting $k = 0$, we have

$$y_1 = a_0 \left[1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \cdots \right] \qquad \ldots (4)$$

Case-(2) : Let $k = 1/2$ and hence (3) assumes the form

$$a_{r+1} = \frac{a_r}{2(r+3/2)} = \frac{a_r}{2r+3} \quad (r \geq 0)$$

Putting $r = 0, 1, 2, 3, \ldots$ we obtain,

$$a_1 = \frac{a_0}{3};\ a_2 = \frac{a_1}{5} = \frac{a_0}{15};\ a_3 = \frac{a_2}{7} = \frac{a_0}{105};\ a_4 = \frac{a_3}{9} = \frac{a_0}{945} \text{ and so-on.}$$

We substitute these values in the expanded form of (2) to obtain,

$$y = a_0 x^k \left[1 + \frac{x}{3} + \frac{x^2}{15} + \frac{x^3}{105} + \frac{x^4}{945} + \cdots \right]$$

Putting $k = 1/2$ we have,

$$y_2 = a_0 \sqrt{x} \left[1 + \frac{x}{3} + \frac{x^2}{15} + \frac{x^3}{105} + \frac{x^4}{945} + \cdots \right] \qquad \ldots (5)$$

The complete solution of (1) is given by

$$y = Ay_1 + By_2$$

Let us denote $Aa_0 = c_1$ and $Ba_0 = c_2$

Thus the required series solution is given by

$$y = c_1 \left[1 + \frac{x}{2 \cdot 1!} + \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} + \frac{x^4}{2^4 \cdot 4!} + \cdots \right]$$
$$+ c_2 \sqrt{x} \left[1 + \frac{x}{3} + \frac{x^2}{3 \cdot 5} + \frac{x^3}{3 \cdot 5 \cdot 7} + \frac{x^4}{3 \cdot 5 \cdot 7 \cdot 9} + \cdots \right]$$

EXERCISES

1. Obtain the series solution of the equation $\frac{dy}{dx} = ky$ and verify with the analytical solution.
2. Solve in series the equation : $(1 - x^2) y' - y = 0$
3. Obtain the power series solution of the equation : $(1 - x^2) y'' - 2xy' + 2y = 0$
4. Use Frobenius method to solve the equation $3xy'' + 2y' + y = 0$
5. Obtain the power series solution of the equation $16 x^2 y'' + 16 xy' + (16 x^2 - 1) y = 0$ by Frobenious method.

ANSWERS

1. $y = a_0 \left[1 + \frac{kx}{1!} + \frac{k^2 x^2}{2!} + \frac{k^3 x^3}{3!} + \cdots \right]$; $y = a_0 e^{kx}$
2. $y = a_0 \left[1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3x^4}{8} + \frac{11x^5}{40} + \cdots \right]$
3. $y = a_0 \left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} - \cdots \right] + a_1 x$
4. $y = c_1 \left[1 + x + \frac{x^2}{4} + \frac{x^3}{4.7} + \cdots \right] = c_2 x^{2/3} \left[1 + \frac{x}{3} + \frac{x^2}{3.6} + \frac{x^3}{3.6.9} + \cdots \right]$
5. $y = c_1 c^{1/4} \left[1 - \frac{x^2}{5} + \frac{x^4}{90} - \cdots \right] + c_2 x^{-1/4} \left[1 - \frac{x^2}{3} + \frac{x^4}{42} - \cdots \right]$